REGULAR BASIS AND FUNCTOR EXT

A THESIS
SUBMITTED TO THE DEPARTMENT OF MATHEMATICS
AND THE INSTITUTE OF ENGINEERING AND SCIENCES
OF BİLKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE

By
Zehra Ertuğrul
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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

REGULAR BASIS AND FUNCTOR EXT

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M.S. in Mathematics
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August 23, 1994

This work is a study of the relation between the vanishing of Ext functor and the existence of regular bases in the cartesian product and tensor product of some special Köthe spaces. We give some new results concerning $S_g$ Spaces in Chapter 3 and the study in the last chapter is about the existence of pseudo-regular bases in the cartesian product and tensor product of two regular Schwartz Köthe spaces $E$ and $F$, one of which having property $(DN)$, when $\text{Ext}(E \times F, E \times F)$ vanishes.

Keywords: Fréchet Space, Nuclear Space, Schwartz Space, Köthe matrix, Köthe Space, Dragilev $(L_f(a,r))$ Space, $S_g(a,r)$ Space, Regular Basis, Pseudo-Regular Basis, Functor Ext, Property HP, Property HP $\not\chi 1$.
ÖZET

DÜZGÜN TABAN VE EXT FUNKTORU

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Matematik Yüksek Lisans
Tez Yöneticisi: Prof. Mefharet Kocatepe
23 Ağustos 1994

Bu tezde iki Köthe uzayının çarpım uzaylarının Ext funktorunun sıfır olması ile çarpım ve tensör çarpım uzaylarının düzgün tabanlarının olması arasındaki ilişki çalışıldı. Üçüncü bölümde \( S_g \) uzayları ele alındı, son bölümde ise biri \( (DN) \) özelliğine sahip iki Köthe uzayının çarpım uzaylarının Ext funktoru sıfır olduğunda çarpım ve tensör çarpım uzaylarının yaklaşıklık- düzgün tabanlarının olduğu gösterildi.

Anahtar Sözcükler: Fréchet Uzayı, Nuclear Uzayı, Schwartz Uzayı, Köthe matrisi, Köthe Uzayı, Dragilev \((L_f(a,r))\) Uzayı, \(S_q(a,r)\) Uzayı, Düzensiz Taban, Yaklaşıklık- Düzensiz Taban, Ext Funktoru, HP, HP \(X'1\).
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Chapter 1

Introduction

The idea of a regular basis in a nuclear Fréchet space was introduced by M. M. Dragilev [2] in 1965. Its importance lies in the study of quasi-equivalence of bases such that every space with a regular basis has the quasi-equivalence property. This property was useful in the computation of various topological linear invariants and in the study of Cartesian Products and as a means of classifying nuclear Köthe spaces.

In the second chapter, we give some definitions and preliminary results which we use in the sequel. In the other chapters, the existence of a regular basis in the Köthe spaces with the vanishing of Ext functor was studied.

Krone [7] has shown that if $\lambda(A)$ is a Köthe space with property $(DN)$ and satisfying the condition $\text{Ext}^1(\lambda(A), \lambda(A)) = 0$ then $\lambda(A)$ always has a regular basis. In her study of this result, Kocatepe [6] has found that the condition $(DN)$ is necessary and she has given an example of a nuclear Köthe space $\lambda(A)$ such that $\text{Ext}^1(\lambda(A), \lambda(A)) = 0$, but $\lambda(A)$ has no regular basis.

Moreover, in the same paper [6] it was stated, with no proof, that if $E$ and $F$ are two Dragilev spaces (defined by functions with comparable growth rates) and $\text{Ext}^1(E \times F, E \times F) = 0$, then $E \times F$ and $E \bigcirc F$ have regular bases.

In this work, first we study the proof of this theorem. Then we consider the situation in $S_p(a,r)$ spaces, which were introduced by V.V. Kashirin
[5], and in the third chapter we give a proof of the same theorem concerning $S_g(a,r)$ spaces.

In [6], Kocatepe has also shown that if $\text{Ext}(\lambda(B), \lambda(A)) = 0$, for two Schwartz regular Köthe spaces with $\lambda(A)$ having property $(DN)$ and $\lambda(B)$ having property $(\Omega)$, then $\lambda(A) \times \lambda(B)$ and $\lambda(A) \hat{\otimes} \lambda(B)$ have regular bases.

In the last chapter, we study the relationship between the vanishing of the functor $\text{Ext}(E \times F, E \times F)$ for two Köthe spaces $E$ and $F$, one of which having property $(DN)$, and the existence of a pseudo-regular basis in $E \times F$ and $E \hat{\otimes} F$. So we drop one of the conditions: $(\Omega)$ in [6] and we get a pseudo-regular basis, which is weaker than regularity (still unknown whether strictly weaker) but is strong enough to obtain almost all of the results that can be obtained using regularity, especially the quasi-equivalence property, [1].
Chapter 2

Definitions and Some Preliminary Results

We shall mean by a locally convex space (l.c.s) a locally convex Hausdorff space and by a Fréchet space we mean a complete metrizable l.c.s. N denotes the set of natural numbers and \( \mathbb{R} \) the set of real numbers.

2.1 Nuclear and Schwartz Spaces

For any two normed linear spaces \( E \) and \( F \), a continuous linear map \( T : E \to F \) is called nuclear if there exist continuous linear forms \( (u_n) \in E' \) and \( (y_n) \in F \) such that

\[
\begin{align*}
(i) & \quad \sum_{n=1}^{\infty} \|u_n\| \|y_n\| < \infty \\
(ii) & \quad \forall x \in E, \quad Tx = \sum_{n=1}^{\infty} u_n(x)y_n
\end{align*}
\]

\( T \) is called precompact if \( T(U) \) is a precompact (\( \overline{T(U)} \) is compact) subset of \( F \), where \( U \) is the closed unit ball in \( E \).

Let \( E \) be a l.c.s. \( U(E) \) is a base of all absolutely convex, closed neighborhoods. Let \( U \in U(E) \) and \( N(U) = p_U^{-1}(o) = \bigcap_{n=1}^{\infty} \frac{1}{n} U \) where \( p_U(.) \) denotes the Minkowski functional of \( U \). Since \( p_U \) is continuous, \( N(U) \) is a
closed subspace of $E$. Let $E(U)$ be the quotient space $E/N(U)$ normed by $\|x(U)\| = pu(x)$ where

$$x(U) = \{ y \in E : x - y \in N(U) \} = \{ y \in E : pu(x - y) = 0 \} = K(U)x$$

and $K(U) : E \rightarrow E/N(U)$, $K(U)x = x(U)$. If $V \in U(E), V \subset U$, then $N(V) \subset N(U)$. Define $K(V, U) : E(V) \rightarrow E(U)$ by $K(V, U)x(V) = x(U)$. $K(V, U)$ is well-defined, linear and continuous.

A l.c.s. $E$ is called a **Nuclear Space** if it satisfies

$$\forall U \in U(E) \ \exists V \in U(E), V \subset U \text{ such that } K(V, U) \text{ is nuclear.}$$

$E$ is called **Schwartz Space** if it satisfies

$$\forall U \in U(E) \ \exists V \in U(E), V \subset U \text{ such that } K(V, U) \text{ is precompact.}$$

### 2.2 Bases and Basic Definitions

A sequence of elements $(x_n)$ in a l.c.s. $E$ is called a **basis** if for each element $x \in E$ there is a uniquely determined sequence of scalars $(t_n)$ such that $x = \lim_{n \to \infty} \sum_{i=1}^{n} t_i x_i$. Two bases $(x_n)$ and $(y_n)$ in a nuclear Fréchet space $E$ are **equivalent** if $\sum t_n x_n$ converges in $E$ iff $\sum t_n y_n$ does. The bases are **semi-equivalent** if there is a sequence of positive numbers $(t_n)$ such that $(t_n x_n)$ is equivalent to $(y_n)$. They are **quasi-equivalent** if there is a rearrangement of one which is semi-equivalent to the other. A nuclear Fréchet space with a basis has the **quasi-equivalence property** if all bases are quasi-equivalent.

A representation of a basis $(x_n)$ in a nuclear Fréchet space $E$ is an infinite matrix $(a_n^k)$ for which there exists a fundamental sequence of seminorms $(p_k)$ defining the topology of $E$ such that $a_n^k = p_k(x_n)$. The basis is **regular** if it has a representation $(a_n^k)$ such that

$$\frac{a_{n+1}^k}{a_n^k} \leq \frac{a_{n+1}^{k+1}}{a_n^{k+1}} \text{ for all } k, n.$$ 

In order to show that a basis is not regular, one must check all possible representations. So, to avoid this problem a property weaker than regularity was introduced in [1].
A basis \((x_n)\) in a nuclear Fréchet space is pseudo-regular if it has a representation \((a_n^k)\) such that

\[
\forall p \quad \exists q \quad \text{st.} \quad \forall r > q \quad \exists s > p \quad \text{and} \quad M > 0 \quad \frac{a_n^r}{a_n^q} \leq M \frac{a_n^s}{a_n^p}, \quad \forall m \leq n.
\]

If this holds for a given representation of \((x_n)\) then it holds for every representation of \((x_n)\) and the converse is also true.

It is clear that every regular basis is pseudo-regular and it was shown in [1], Thm. 1, that every nuclear Fréchet space with a pseudo-regular basis has the quasi-equivalence property.

### 2.3 Köthe and Power Series Spaces

A matrix \(A = (a_n^k)\) of non-negative scalars satisfying

(i) \(\forall n, k \quad a_n^k \leq a_n^{k+1}\)

(ii) \(\forall n \quad \sup_k a_n^k > 0\)

is called a Köthe matrix and the sequence space

\[
\lambda(A) = \{ (\xi_n) : \| (\xi_n) \|_k = \sum_n |\xi_n| a_n^k < \infty \quad \forall k \}
\]

topologized by the seminorms \(||.||_k\) is called a Köthe space. \(\lambda(A)\) is a complete l.c.s.

Let \(\alpha = (\alpha_n)\) be a non-decreasing sequence of positive numbers. Consider the Köthe set

\[
A = \{ (e^{k\alpha_n}) : k \in \mathbb{N} \}
\]

Then \(\lambda(A) = \lambda_{\infty}(\alpha)\) is called a power series space of infinite type.

The finite type power series space \(\lambda_r(\alpha), 0 < r < \infty\), generated by \(\alpha\) is the sequence space \(\lambda(A)\) where

\[
A = \{ (r_k)^{\alpha_n} : k \in \mathbb{N} \}, \quad r_k \not\in r.
\]

The proof of the following nuclearity criterion, known as Grothendieck-Pietsch Criterion, for Köthe spaces has been given in [11] 6.1.2:
Theorem 1: A Köthe space $\lambda(A)$ is nuclear iff
$$\forall k \ \exists l \ \text{such that} \ \left(\frac{a^k}{a^l}\right) \in l^1, \ \text{i.e.} \ \sum_{n} a^k_n/a^l_n < \infty.$$

Theorem 2: A Köthe space $\lambda(A)$ is Schwartz iff
$$\forall k \ \lim_{n} \frac{a^k_n}{a^{k+1}_n} = 0, \ 0 < a^k_n < a^{k+1}_n.$$

A Köthe matrix $A = (a^k_n)$ and also the Köthe space $\lambda(A)$ are called of type $(d_i), i = 1, 2, 5$, if the corresponding conditions below are satisfied:

$$(d_1) \ \exists k \ \forall j \ \exists l \ \sup_{n} \frac{(a^j_n)^2}{a^k_n a^l_n} < \infty$$

$$(d_2) \ \forall k \ \exists j \ \forall l \ \sup_{n} \frac{a^k_n a^j_n}{(a^l_n)^2} < \infty$$

$$(d_5) \ \exists M \geq 1 \ \forall k, n \ \frac{a^{k+1}_n}{a^k_n} \leq \left(\frac{a^{k+2}_n}{a^{k+1}_n}\right)^M$$

The conditions $(d_1)$ and $(d_2)$ were introduced by M. M. Dragilev in [2], and $(d_5)$ by E. Dubinsky in [3].

A basis $(x_n)$ of a nuclear Fréchet space $E$ is said to be of type $(d_i), i = 1, 2, 5$, if there exists a fundamental system of norms $(\| \cdot \|_k)$ such that $a^k_n = \|x_n\|_k$ is of type $(d_i), i = 1, 2, 5$.

A Fréchet space $E$ is said to have the property $(DN)$ if
$$\exists k \ \forall j \ \exists l, C > 0 \ \| \cdot \|_j^2 \leq C \| \cdot \|_k \| \cdot \|_l$$

$$(\tilde{\Omega}) \ \text{if} \ \forall p \ \exists q \ \forall k \ \exists C > 0 \ (\| \cdot \|_q)^2 \leq C \| \cdot \|_p \| \cdot \|_p^*$$

The property $(DN)$ was introduced by D. Vogt in [13] and $(\tilde{\Omega})$ by M. J. Wagner in [14] to characterize subspaces and quotients of nuclear and stable power series spaces of infinite type and finite type, respectively. If a nuclear Fréchet space has a basis, then $(d_1)$ and $(DN)$ are equivalent, $(d_2)$ is equivalent to $(\tilde{\Omega})$.

If $E, F$ are nuclear Fréchet spaces with bases $(x_n), (y_n)$, respectively, then the completed topological tensor product $E \hat{\otimes} F$ is a nuclear Fréchet space with basis $(x_m \otimes y_n)_{(m,n) \in \mathbb{N} \times \mathbb{N}}$. If $(a^k_n), (b^k_n)$ are matrix representations of $(x_n), (y_n)$, respectively, then $(a^k_m b^k_n)$ is a matrix representation of $(x_m \otimes y_n)$. 

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2.4 The Functor Ext

By an exact sequence of Fréchet spaces $E, F$ and $G$ we mean a sequence

$$0 \to E \xrightarrow{i} F \xrightarrow{q} G \to 0$$

where $i$ is an embedding of $E$ into $F$, $q$ a continuous surjective linear mapping from $F$ onto $G$ and $i(E) = \text{Ker}(q)$. We say that the sequence splits if there exists a continuous right inverse of $q$ or equivalently a continuous left inverse of $i$, which means that $F$ is isomorphic to the direct sum of $E$ and $G$.

The definition for Ext functor below was taken from Vogt [12], so for the details we refer the reader to [12].

Let $\mathcal{F}$ be the category of Fréchet spaces, $\mathcal{L}$ the category of linear spaces, $L(E, F)$ the linear space of continuous linear maps from $E$ to $F$. A space $I$ in $\mathcal{F}$ is called injective iff for each $E_1$ in $\mathcal{F}$, each closed subspace $E_0 \subset E_1$ and each $\varphi \in L(E_0, I)$ there exists an extension $\phi \in L(E_1, I)$.

An injective resolution of $F$ is an exact sequence

$$0 \to F \to I_0 \xrightarrow{i_0} I_1 \xrightarrow{i_1} I_2 \to \ldots$$

where $I_k$ is injective for all $k$. Every $F \in \mathcal{F}$ has an injective resolution.

We denote by $\text{Ext}^k(E, \cdot)$ the right derived functors of the functor $L(E, \cdot)$ acting from $\mathcal{F}$ to $\mathcal{L}$. So for any injective resolution (1) of $F$ we have

$$\text{Ext}^k(E, F) \cong \ker j_k / \text{im} j_{k-1} \quad k = 1, 2, \ldots$$

where $j_k : L(E, I_k) \to L(E, I_{k+1})$ is defined by $j_k(A) = i_k \circ A$ for $A \in L(E, I_k)$ and $\text{Ext}^0(E, F) = L(E, F)$.

Vogt has proven the following theorem in [12], Theorem 1.8., hence we only give the statement of it:
**Theorem:** The following are equivalent

1. \( \text{Ext}^1(E, F) = 0 \)
2. For each exact sequence \( 0 \to F \to G \xrightarrow{f} H \to 0 \) and \( \varphi \in L(E, H) \), there exists a lifting \( \psi \in L(E, G) \), i.e. a map \( \psi \) with \( \varphi = q \circ \psi \).
3. Each exact sequence \( 0 \to F \to G \to E \to 0 \) splits.
4. For each sequence \( 0 \to H \xrightarrow{f} G \to E \to 0 \) and \( \varphi \in L(H, F) \), there exists an extension \( \phi \in L(G, F) \), i.e. a map \( \phi \) with \( \varphi = \phi \circ i \).

We say that, for two Köthe spaces \( E = \lambda(a_k^i) \) and \( F = \lambda(b_k^j) \), the pair \( (E, F) \) satisfies \((S_1^*)\), respectively \((S^*)\), and write this as \( (E, F) \in (S_1^*) \) if the corresponding condition is fulfilled:

\[(S_1^*) \quad \exists n_0 \quad \forall \mu \quad \exists k \quad \forall K, m \quad \exists n, S > 0 \quad \forall i, j : \]
\[\frac{a_i^m}{b_j^k} \leq S \max \left\{ \frac{a_i^n}{b_j^k}, \frac{a_i^{n_0}}{b_j^k} \right\} \]

\[(S^*) \quad \forall \mu \quad \exists n_0, k \quad \forall K, m \quad \exists n, S > 0 \quad \forall i, j : \]
\[\frac{a_i^m}{b_j^k} \leq S \max \left\{ \frac{a_i^n}{b_j^k}, \frac{a_i^{n_0}}{b_j^k} \right\} \]

The condition \((S^*)\) has been rewritten in an apparently strengthened form in [8], Lemma 1.2. Namely, if \( (E, F) \in (S^*) \) then either \( E = l^1 \) or \( (E, F) \) satisfies the following condition:

\[(S^*)_o \quad \forall \mu \quad \exists n_0, k \quad \forall K, m, R > 0 \quad \exists n, S > 0 \quad \forall i, j : \]
\[\frac{a_i^m}{b_j^k} \leq \max \left\{ S \frac{a_i^n}{b_j^k}, \frac{1}{R} \frac{a_i^{n_0}}{b_j^k} \right\} \]

So, \( (E, F) \in (S^*)_o \) is equivalent to

\[(S^*)_1 \quad \forall \mu \quad \exists n_0, k \quad \forall K, m, R > 0 \quad \exists \tilde{n}, \tilde{S} > 0 \quad \forall i, j : \]
\[\frac{a_i^m}{b_j^k} \leq \max \left\{ S \frac{a_i^{n_0}}{b_j^k}, \frac{1}{R} \frac{a_i^n}{b_j^k} \right\} \]

We shall also use \( (F, E) \in (S^*)_o \), which we write as

\[(S^*)_2 \quad \forall \bar{\mu} \quad \exists \bar{n}_0, \bar{k} \quad \forall K, m, R > 0 \quad \exists \bar{n}, \bar{S} > 0 \quad \forall i, j : \]
\[\frac{b_j^m}{a_i^k} \leq \max \left\{ S \frac{b_j^n}{a_i^k}, \frac{1}{R} \frac{b_j^{n_0}}{a_i^k} \right\} \]

We need this equivalent forms of \((S^*)\) in the last chapter of this thesis.

To simplify the notation we shall write \( \text{Ext}(E, F) \) for \( \text{Ext}^1(E, F) \) and \( \text{Ext}(E) \) for \( \text{Ext}(E, E) \). The following results are due to Vogt [12]:
(a) \( \text{Ext}(E, F) = 0 \) iff every exact sequence \( 0 \to F \to G \to E \to 0 \) of Fréchet spaces and continuous linear maps splits.

(b) \((S^*_1) \Rightarrow \text{Ext}(E, F) = 0 \Rightarrow (S^*)\)

In [8], Krone-Vogt have shown that if \( E \) and \( F \) are both Köthe spaces then \( \text{Ext}(E, F) = 0 \) iff \( (E, F) \in (S^*) \).

We also know that if \( \text{Ext}(E \times F, E \times F) = 0 \), then we have \( \text{Ext}(E) = \text{Ext}(F) = \text{Ext}(E, F) = \text{Ext}(F, E) = 0 \).

2.5 \( L_f(a,r) \) Spaces

\( L_f(a,r) \) spaces, also called Dragilev spaces, were introduced by M. M. Dragilev in [2] in 1965.

**Definition**: Let \( f : \mathbb{R} \to \mathbb{R} \) be an odd, increasing and logarithmically convex function (i.e. \( \log f \circ \exp \) is convex on \([0, \infty)\)), which is called Dragilev function. Let \( a = (a_n) \) be an increasing sequence of positive numbers such that \( \lim_n a_n = \infty \) and \( r_k \) a strictly increasing sequence of real numbers with \( \lim_k r_k = r \) where \(-\infty < r \leq +\infty\). Then the space \( L_f(a,r) \) is defined as the Köthe space \( \lambda(A) \) where \( A = (a^k_n) = e^{r_k a_n} \).

From logarithmic convexity of \( f \) it follows that \( \tau(a) = \lim_{x \to \infty} \frac{f(ax)}{f(x)} \) exists for every \( a > 1 \). Moreover either \( \tau(a) = \infty \) for all \( a > 1 \) or \( \tau(a) < \infty \) for all \( a > 1 \).

If \( \tau(a) = \infty \) for all \( a > 1 \), then \( f \) is called rapidly increasing.

If \( \tau(a) < \infty \) for all \( a > 1 \), then \( f \) is called slowly increasing.

**Basic Properties**: The following properties of Dragilev spaces are either immediate or have been proven by M. M. Dragilev [2].

(a) \( L_f(a,r) \) is regular.

(b) \( L_f(a,r) \) is Schwartz and nuclear if \( \forall k \exists l \sum_j \frac{e^{f(r_k a_j)}}{e^{f(r_l a_j)}} < \infty \).
(c) $L_f(a, r)$ is isomorphic to a power series space iff $f$ is slowly increasing, in which without loss of generality we can take $f$ as the identity function. When $f$ is slowly increasing we have $L_f(a, \infty) \cong \lambda_\infty(\alpha)$ and $L_f(a, r) \cong \lambda_1(\alpha)$, $r < \infty$.

(d) For a Dragilev function $f$,

(i) $L_f(a, r) \cong L_f(a, 1)$ if $0 < r < \infty$.

(ii) $L_f(a, r) \cong L_f(a, -1)$ if $r < 0$.

Hence basically there are four types of Dragilev spaces:

$r = -1, r = 0, r = 1, r = \infty$.

(e) If $f$ is rapidly increasing, then

$L_f(a, -1)$ and $L_f(a, 0)$ are of type $(d_2)$, $(\Omega)$.

$L_f(a, 1)$ and $L_f(a, \infty)$ are of type $(d_1)$, $(DN)$.

2.6 $S_g(a, r)$ Spaces

$S_g(a, r)$ spaces were introduced by V. V. Kashirin [5] in 1980 to refute a conjecture of M. M. Dragilev who had asked whether every regular $(d_1)$ or $(d_2)$ type nuclear Köthe space was isomorphic to some Dragilev space.

An $S_g(a, r)$ space is defined similar to an $L_f(a, r)$ with the exception that $f$ is a convex function on $[0, \infty)$ instead of logarithmically convex. Hence rapidly or slowly increasing has no meaning anymore.

Basic Properties: These are either immediate or the proofs can be found in [9].

(a) $S_g(a, r)$ is regular.

(b) If $r < 0$ then $S_g(a, r) \cong S_g(b, -1)$ and is of type $(d_2)$.

(c) If $r = 0$ then $S_g(a, r)$ is $(d_2)$.

(d) If $0 < r < \infty$ then $S_g(a, r) \cong S_g(b, 1)$ and is of type $(d_5)$.

(e) If $r = +\infty$ then $S_g(a, r)$ is $(d_1)$.

(f) Every $L_f(a, r)$ is an $S_g(a, r)$ for some $g$, but the converse is false.
Moreover we have the following properties for an odd, increasing, convex function \( g : \quad \forall x \geq y \geq 0 \):

\[
\begin{align*}
(1) \quad & \forall x \geq \hat{x} \quad \forall y \geq \hat{y} \quad \frac{g(\hat{x}) - g(\hat{y})}{\hat{x} - \hat{y}} \leq \frac{g(x) - g(y)}{x - y} \\
(ii) \quad & g(x) + g(y) \leq g(x + y) \\
(iii) \quad & g(x) - g(y) \geq g(x - y) \\
(iv) \quad & g(0) = 0 \\
(v) \quad & \text{for } c > 1, \ cg(x) \leq g(cx).
\end{align*}
\]

In his thesis [4], J. Hebbecker has considered a pair of Dragilev spaces \( E \) and \( F \) and has obtained the necessary and sufficient conditions for \((S^*_1)\) and \((S^*)\). He has used the following conditions in his complete characterizations:

Let \( a = (a_n) \) and \( b = (b_n) \) be two exponent sequences. We say

\[
(a, b) \in HP \quad \text{if the set of finite limit points of the set } \left\{ \frac{a_i}{b_j} : i, j \in \mathbb{N} \right\}
\]

is bounded,

\[
(a, b) \in HP \setminus \{0\} \quad \text{if there is } c \in [0,1) \text{ such that the set of finite limit points of the set } \left\{ \frac{a_i}{b_j} : i, j \in \mathbb{N} \right\} \text{ is contained in } [0, c] \cup [1, \infty).
\]

For simplicity, we use the notation \( LIM\left\{ \frac{a}{b} \right\} \) for the set of the finite limit points of the set \( \left\{ \frac{a_i}{b_j} : i, j \in \mathbb{N} \right\} \). In [10], Lemma 5.1., K. Nyberg has proven the following:

\( LIM\left\{ \frac{a}{b} \right\} \) is bounded if and only if there exist strictly increasing sequences of indices \( (m_i) \) and \( (n_i) \) such that

\[
(i) \quad \sup_{i} \frac{a_{m_i+1}}{b_{n_i+1}} < \infty \quad \text{and} \quad (ii) \quad \lim_{i \to \infty} \frac{a_{m_i+1}}{b_{n_i}} = \infty.
\]

Using exactly the same argument, we prove the following lemma:
Lemma: \((a, b) \in HP \not\subset 1\) if and only if there exist strictly increasing sequences of indices \((m_i)\) and \((n_i)\) such that

\[
(i) \quad \lim_{i \to \infty} \sup_n \frac{a_{m_{i+1}}}{b_{n_{i+1}}} < 1 \quad \text{and} \quad (ii) \quad \lim_{i \to \infty} \inf_n \frac{a_{m_{i+1}}}{b_{n_{i}}} \geq 1
\]

Proof: By definition \((a, b) \in HP \not\subset 1\) if and only if there exist \(c \in [0, 1)\) such that \(LIM \{ \frac{a}{b} \} \subset [0, c] \cup [1, \infty)\). Without loss of generality we may assume \(\frac{a_1}{b_1} \leq c < 1\). We choose

\[
m_i = \max\{m : a_m \leq cb_1\} \quad \text{and} \quad n_i = \max\{n : cb_n < a_{m_{i+1}}\}
\]

Suppose \(m_i\) and \(n_i\) have been chosen in such a way that

\[
m_i = \max\{m : a_m \leq cb_{n_{i-1}+1}\} \quad \text{and} \quad n_i = \max\{n : cb_n < a_{m_{i+1}}\}
\]

then \(cb_{n_{i+1}} \geq m_i + 1\); so choose

\[
m_{i+1} = \max\{m : a_m \leq cb_{n_{i+1}}\} \geq m_i + 1
\]

now we can take

\[
n_{i+1} = \max\{n : cb_n < a_{m_{i+1}}\} \geq n_i + 1
\]

Hence by induction we construct subsequences \((m_i)\) and \((n_i)\) of \(\mathbb{N}\) such that

\[
\frac{a_{m_{i+1}}}{b_{n_{i+1}}} \leq c < 1 \quad \text{and} \quad \frac{a_{m_{i+1}}}{b_{n_{i}}} > c
\]

which means

\[
\lim i \frac{a_{m_{i+1}}}{b_{n_{i+1}}} \in [0, c] \quad \text{and} \quad \lim i \frac{a_{m_{i+1}}}{b_{n_{i}}} \in [1, \infty)
\]

then it follows that

\[
\lim_{i} \sup_{n} \frac{a_{m_{i+1}}}{b_{n_{i+1}}} < 1 \quad \text{and} \quad \lim_{i} \inf_{n} \frac{a_{m_{i+1}}}{b_{n_{i}}} \geq 1
\]

Conversely, let \(c = \lim_{i} \sup_{n} \frac{a_{m_{i+1}}}{b_{n_{i+1}}} < 1\) and if possible \(p\) be a finite limit point for some subsequences \((\mu_k)\) and \((\nu_k)\) of \(\mathbb{N}\) such that \(c < p < 1\), i.e. \(\lim_{k \to \infty} \frac{a_{\mu_k}}{b_{\nu_k}} = p\). When \(\frac{a_{\mu_k}}{b_{\nu_k}} > c\) i.e. \(k\) is sufficiently large and \(m_i < \mu_k \leq m_{i+1}\) we have \(\nu_k \leq n_i\). So \(\frac{a_{\mu_k}}{b_{\nu_k}} \geq \frac{a_{m_{i+1}}}{b_{n_i}}\). Since \(\lim_{i} \inf_{n} \frac{a_{m_{i+1}}}{b_{n_{i}}} \geq 1\), we get \(p \geq 1\) which is a contradiction. Therefore \(LIM \{ \frac{a}{b} \} \subset [0, c] \cup [1, \infty)\).
In the second part of [4], he considers pairs of Dragilev spaces which are defined by rapidly increasing Dragilev functions $f$ and $g$. He defines

$$f \succ g \quad \text{if} \quad g^{-1} \circ f \text{ is a rapidly increasing function}$$

$$f \prec g \quad \text{if} \quad f^{-1} \circ g \text{ is a rapidly increasing function}$$

$$f \approx g \quad \text{if} \quad g^{-1} \circ f \text{ and } f^{-1} \circ g \text{ are both logarithmically convex and slowly increasing.}$$

In our study of $S_g(a,r)$ spaces we define the relations between $f$ and $g$ as follows:

$$f \succ g \quad \text{if} \quad \forall a > 0 \lim_{x \to \infty} \frac{f^{-1} g(ax)}{f^{-1} g(x)} = 1$$

$$f \prec g \quad \text{if} \quad \forall a > 0 \lim_{x \to \infty} \frac{g^{-1} f(ax)}{g^{-1} f(x)} = 1$$

$$f \approx g \quad \text{if} \quad g^{-1} \circ f \text{ and } f^{-1} \circ g \text{ are both convex, in other words there exist } \alpha \text{ and } \lambda > 0, \text{ such that } g^{-1} f(x) = \lambda x + \alpha \quad \forall x \in \mathbb{R}. \text{ But properties of a convex function } (iv) \text{ gives } \alpha = 0 \text{ so } f(x) = g(\lambda x).$$

We note that the conditions $\lim_{x \to \infty} \frac{f^{-1} g(ax)}{f^{-1} g(x)} = 1$ for all $a > 1$ and $\lim_{x \to \infty} \frac{g^{-1} f(bx)}{g^{-1} f(x)} = \infty$ for all $b > 1$ are equivalent. But in our considerations, we are not assuming that $f$ and $g$ are rapidly increasing.

Using this definitions and the vanishing of Ext functor between two $S_g(a,r)$ spaces, we try to find similar conditions on $(S^*)$ as in the thesis of Hebbecker, with one exception: we could not find a condition to the case $(S_f(a,-1), S_g(b,1))$ when $f \approx g$. Then we search the regularity in the cross and tensor products of the spaces.
Chapter 3

Some Results On The Pair \((S_f(a, r), S_g(b, s))\)

In this chapter, first we give the proof of

**Theorem 1:** Let \(E = S_f(a, r),\) \(F = S_g(b, s)\) and assume that \(\text{Ext}(E \times F, E \times F) = 0.\) Then \(E \times F\) has a regular basis in the following cases

(i) \(f \prec g\) or \(f \succ g\).

(ii) \(f \approx g\) and \(rs = -1.\)

### 3.1 \((S_f(a, r), S_g(b, s))\) \((0 < r, s \leq \infty)\)

#### 3.1.1 \(E = S_f(a, \infty)\) \(F = S_g(b, \infty)\)

**Lemma 1:**

(a) If \((S_f(a, \infty), S_g(b, \infty)) \in (S^*),\) then

\[
\exists k \ \forall K, m \ \exists n, i_o, j_o \ \forall i \geq i_o, \forall j \geq j_o:
\]

\[
g(s_kb_j) < f(rma_i) \implies g(s_kb_j) \leq f(rma_i).
\]

(b) If \((S_g(b, \infty), S_f(a, \infty)) \in (S^*),\) then

\[
\exists k \ \forall K, m \ \exists n, i_o, j_o \ \forall i \geq i_o, \forall j \geq j_o:
\]

\[
f(rka_i) < g(smb_j) \implies f(rka_i) \leq g(smb_j).
\]
Proof: We only give the proof of (a), the other case is symmetrical.

(a) We may choose $2r_k \leq r_{k+1}$ and $2s_k \leq s_{k+1}$, since $r_k, s_k \not\to \infty$.

$(S_f(a, \infty), S_g(b, \infty)) \in (S^*)$ implies

$$f(r_m a_i) - g(s_k b_j) \leq S + \max\{f(r_n a_i) - g(s_k b_j), f(r_n a_i) - g(s_k b_j)\}$$

So we have

$$f(r_m a_i) - f(r_n a_i) \leq S + g(s_k b_j) - g(s_k b_j)$$

and/or

$$g(s_k b_j) - g(s_k b_j) \leq S + f(r_n a_i) - f(r_n a_i)$$

We can find $i_1, j_1 \in \mathbb{N}$ such that for all $i \geq i_1$, for all $j \geq j_1$:

$S + g(s_k b_j) - g(s_k b_j) \leq g(s_k b_j)$ and $S + f(r_n a_i) - f(r_n a_i) \leq f(r_n a_i)$

Since $f$ and $g$ are convex, for all $K > k$ and $m > n_o$ we have

$$\frac{1}{2} \leq \frac{s_k - s_k}{s_k} \leq \frac{g(s_k b_j) - g(s_k b_j)}{g(s_k b_j)}$$

and

$$\frac{1}{2} \leq \frac{r_m - r_{n_o}}{r_m} \leq \frac{f(r_m a_i) - f(r_{n_o} a_i)}{f(r_m a_i)}$$

Thus

$$\frac{1}{2} g(s_k b_j) \leq g(s_k b_j) - g(s_k b_j), \text{ and } \frac{1}{2} f(r_m a_i) \leq f(r_m a_i) - f(r_{n_o} a_i)$$

Then $(S_f(a, \infty), S_g(b, \infty)) \in (S^*)$ implies

$$\exists k \forall K, m \exists n, i_1, j_1 \forall i \geq i_1, \forall j \geq j_1 :$$

$$\frac{1}{2} g(s_k b_j) \leq f(r_n a_i) \text{ and/or } \frac{1}{2} f(r_m a_i) \leq g(s_k b_j)$$

We can now find $i_2, j_2 \in \mathbb{N}$ such that for all $i \geq i_2$, for all $j \geq j_2$:

$$2f(r_n a_i) \leq f(r_{n+1} a_i) \text{ and } 2g(s_k b_j) \leq g(s_{k+1} b_j)$$

So taking $i_o = \max(i_1, i_2)$, $j_o = \max(j_1, j_2)$ we get

$$\exists k \forall K, m \exists n, i_o, j_o \forall i \geq i_o, \forall j \geq j_o :$$

$$g(s_k b_j) \leq f(r_m a_i) \Rightarrow g(s_k b_j) \leq f(r_n a_i)$$
Proof of the Theorem:

(1) \( f \prec g : (S_f(a, \infty), S_g(b, \infty)) \in (S^*) \Rightarrow (g^{-1}f(a), b) \in H_P. \)

Using Lemma 1 (a),

\[
\exists k \quad \forall K, m \quad \exists n, i_0, j_0 \quad \forall i \geq i_0, \forall j \geq j_0:
\]

\[
g(s_k b_j) < f(r_m a_i) \quad \Rightarrow \quad g(s_k b_j) \leq f(r_m a_i)
\]

So

\[
s_k \frac{g^{-1}f(a_i)}{g^{-1}f(r_m a_i)} < \frac{g^{-1}f(a_i)}{b_j} \quad \Rightarrow \quad s_k \frac{g^{-1}f(a_i)}{g^{-1}f(r_m a_i)} \leq \frac{g^{-1}f(a_i)}{b_j}
\]

Since \( f \prec g, \lim_{i \to \infty} \frac{g^{-1}f(a_i)}{b_j} = 1 \) for all \( m \).

Let \( A \) be a finite limit point of \( \{ \frac{g^{-1}f(a_i)}{b_j} : i, j \in \mathbb{N} \} \), then

\[
\exists k \quad \forall K \quad s_k \leq A \quad \Rightarrow \quad s_K \leq A
\]

which is a contradiction, since \( s_K \not\to \infty \). Hence \( \lim \{ \frac{g^{-1}f(a_i)}{b_j} : i, j \in \mathbb{N} \} \) is bounded, i.e. \( (g^{-1}f(a), b) \in H_P \). We know from the previous chapter, this condition is equivalent to

\[
\exists (m_i), (n_i), \text{ strictly increasing sequences, such that}
\]

(i) \( \sup_i \frac{g^{-1}f(a_{m_i+1})}{b_{n_i+1}} < \infty \) and (ii) \( \lim_{i \to \infty} \frac{g^{-1}f(a_{m_i+1})}{b_{n_i}} = \infty \)

If \( (x_i), (y_i) \) denote the canonical bases in \( E, F \) respectively, we claim that

\[
\cdots, y_{n_i-1+1}, \cdots, y_{n_i}, x_{m_i+1}, \cdots, x_{m_i+1}, y_{n_i+1}, \cdots \quad (1)
\]

is a regular basis for \( E \times F \).

First we show that

\[
\frac{e(g(s_{k+1}b_{n_i}))}{e(g(s_kb_{n_i}))} \leq \frac{e(f(r_{k+1}a_{m_i+1}))}{e(f(r_ka_{m_i+1}))}
\]

i.e. \( g(s_{k+1}b_{n_i}) - g(s_kb_{n_i}) \leq f(r_{k+1}a_{m_i+1}) - f(r_ka_{m_i+1}) \). By (ii) we have,

\[
\forall k \quad \exists i_0 \quad \forall i \geq i_0 \quad g^{-1}f(a_{m_i+1}) \geq s_{k+1}b_{n_i}.
\]

Hence, since \( r_k \not\to \infty \), for large \( k \) we have

\[
g(s_{k+1}b_{n_i}) \leq f(a_{m_i+1}) \leq f(r_ka_{m_i+1}) \quad (\ast)
\]
Therefore
\[ g(s_{k+1}b_{n_i}) - g(s_kb_{n_i}) \leq g(s_{k+1}b_{n_i}) \]
\[ \leq f(r_{k+1}a_{m_{i+1}}) \]
\[ \leq f(r_{k+1}a_{m_{i+1}}) - f(r_{k}a_{m_{i+1}}) \]

Secondly we show that
\[ \frac{e(f(r_{k+1}a_{m_{i+1}}))}{e(f(r_{k}a_{m_{i+1}}))} \leq \frac{e(g(s_{k+1}b_{n_i+1}))}{e(g(s_kb_{n_i+1}))} \]

or \[ f(r_{k+1}a_{m_{i+1}}) - f(r_{k}a_{m_{i+1}}) \leq g(s_{k+1}b_{n_i+1}) - g(s_kb_{n_i+1}). \]
By using (i) and \( f \prec g \) we have,
\[ \exists k_o \forall k \geq k_o \exists i_o \forall i \geq i_o \quad \frac{g^{-1}f(r_{k+1}a_{m_{i+1}})g^{-1}f(a_{m_{i+1}})}{b_{n_{i+1}}} \leq s_k \]

This shows that \[ f(r_{k+1}a_{m_{i+1}}) \leq g(s_kb_{n_{i+1}}) \] (**) So
\[ f(r_{k+1}a_{m_{i+1}}) - f(r_{k}a_{m_{i+1}}) \leq g(s_{k+1}b_{n_{i+1}}) - g(s_{k}b_{n_{i+1}}) \]

Therefore (1) is a regular basis for \( E \times F \).

(2) \( f \approx g \): Although this case is known from Krone [7], for the sake of completeness we give its proof here.

By definition, there exists \( \lambda > 0 \) such that \( f(x) = g(\lambda x) \). Now we may take \( r_k = s_k \nearrow \infty \) and find increasing sequences \((m_i), (n_i)\) which satisfy
\[ \ldots \leq b_{n_i} \leq \lambda a_{m_{i+1}} \leq \ldots \leq \lambda a_{m_{i+1}} \leq b_{n_{i+1}} \leq \ldots \]

Then we show, also in this case, that (1) is a regular basis for \( E \times F \).
First observe that \( s_kb_{n_i} \leq r_k\lambda a_{m_{i+1}} \), for all \( k \). Since \( g \) is convex, we have
\[ g(s_{k+1}b_{n_i}) - g(s_kb_{n_i}) \leq \frac{g(r_{k+1}\lambda a_{m_{i+1}}) - g(r_k\lambda a_{m_{i+1}})}{(r_{k+1} - r_k)\lambda a_{m_{i+1}}} \]
\[ g(s_{k+1}b_{n_i}) - g(s_kb_{n_i}) \leq f(r_{k+1}a_{m_{i+1}}) - f(r_{k}a_{m_{i+1}}) \frac{(s_{k+1} - s_k)b_{n_i}}{(r_{k+1} - r_k)\lambda a_{m_{i+1}}} \]

Since \( s_{k+1} - s_k = r_{k+1} - r_k \) and \( \frac{b_{n_i}}{\lambda a_{m_{i+1}}} \leq 1 \), we get
\[ g(s_{k+1}b_{n_i}) - g(s_kb_{n_i}) \leq f(r_{k+1}a_{m_{i+1}}) - f(r_k a_{m_{i+1}}) \]
Next we use the same argument to get
\[
f(r_{k+1}a_{m_{i+1}}) - f(r_k a_{m_{i+1}}) = g(r_{k+1}\lambda a_{m_{i+1}}) - g(r_k \lambda a_{m_{i+1}}) \\
\leq \frac{g(s_{k+1}b_{n_{i+1}}) - g(s_k b_{n_{i+1}})}{(s_{k+1} - s_k) b_{n_{i+1}}} (r_{k+1} - r_k) \lambda a_{m_{i+1}} \\
\leq g(s_{k+1}b_{n_{i+1}}) - g(s_k b_{n_{i+1}})
\]

3.1.2 \quad E = S_f(a, 1) \quad F = S_g(b, \infty)

Lemma 2:
(a) If \((S_f(a, 1), S_g(b, \infty)) \in (S^*)\), then
\[
\exists n_o, k \quad \forall K, m \quad \exists n, i_o, j_o \quad \forall i \geq i_o, \forall j \geq j_o : \\
g(s_k b_j) < f((r_m - r_n) a_i) \Rightarrow g(s_K b_j) \leq f(2r_n a_i).
\]
(b) If \((S_g(b, \infty), S_f(a, 1)) \in (S^*)\), then
\[
\exists n_o, k \quad \forall K, m \quad \exists n, i_o, j_o \quad \forall i \geq i_o, \forall j \geq j_o : \\
f(r_k a_i) < g((s_m - s_n) b_j) \Rightarrow f(r_K a_i) \leq g(2s_n b_j).
\]

Proof:
(a) \((S_f(a, 1), S_g(b, \infty)) \in (S^*)\) implies
\[
\forall \mu \quad \exists n_o, k \quad \forall K, m \quad \exists n, S > 0 \quad \forall i, j : \\
f(r_m a_i) - f(r_n a_i) \leq S + g(s_k b_j) - g(s_\mu b_j)
\]
and/or
\[
g(s_K b_j) - g(s_k b_j) \leq S + f(r_n a_i) - f(r_m a_i)
\]
Find \(i_o, j_o \in \mathbb{N}\) such that
\[
S + g(s_k b_j) - g(s_\mu b_j) \leq g(s_k b_j)
\]
and/or
\[
S + f(r_n a_i) - f(r_m a_i) \leq f(r_n a_i)
\]
Then
\[
\forall \mu \quad \exists n_o, k \quad \forall K, m \quad \exists n, i_o, j_o \quad \forall i \geq i_o, \forall j \geq j_o
\]
\[ f(r_m a_i) - f(r_n a_i) \leq g(s_k b_j) \text{ and/or } g(s_K b_j) - g(s_k b_j) \leq f(r_n a_i) \]

Then \( (S_f(a, 1), S_g(b, \infty)) \in (S^*) \) \( \Rightarrow \)

\[ \exists n_o, k \quad \forall K, m \quad \exists n, i_o, j_o \quad \forall i \geq i_o, \forall j \geq j_o \]

\[ g(s_k b_j) < f((r_m - r_{n_o})a_i) \Rightarrow g(s_k b_j) \leq f(r_n a_i) + g(s_k b_j) \]

So

\[ \exists n_o, k \quad \forall K, m \quad \exists n, i_o, j_o \quad \forall i \geq i_o, \forall j \geq j_o \]

\[ g(s_k b_j) < f((r_m - r_{n_o})a_i) \Rightarrow g(s_k b_j) \leq f(2r_n a_i) \]

(b) We may prove in a similar way.

**Proof of the Theorem:**

(1) \( f \prec g : (S_f(a, 1), S_g(b, \infty)) \in (S^*) \Rightarrow (g^{-1} f(a), b) \in HP. \)

We use Lemma 2 (a) and a way similar to the proof of the theorem in (1), section 3.1.1. Then the same equivalent condition holds and we claim also in this case, that (1) is a regular basis for \( E \times F. \)

First from (ii) and \( f \prec g, \)

\[ \exists i_o \quad \forall i \geq i_o \text{ and } \forall k, \quad \frac{g^{-1} f(a_{m_{i+1}})}{b_{n_{i}}} \frac{g^{-1} f((r_{k+1} - r_k)a_{m_{i+1}})}{g^{-1} f(a_{m_{i+1}})} \geq s_{k+1} \]

So we get

\[ g(s_{k+1} b_{n_i}) - g(s_k b_{n_i}) \leq g(s_{k+1} b_{n_i}) \leq f((r_{k+1} - r_k)a_{m_i}) \leq f(r_{k+1} a_{m_i}) - f(r_k a_{m_i}) \]

Next consider (i) and \( f \prec g \)

\[ \exists i_o, k_o \quad \forall i \geq i_o \forall k \geq k_o, \quad \frac{g^{-1} f(a_{m_{i+1}})}{b_{n_{i+1}}} \frac{g^{-1} f(r_{k+1} a_{m_{i+1}})}{g^{-1} f(a_{m_{i+1}})} < s_k \]

which shows \( f(r_{k+1} a_{m_{i+1}}) \leq g(s_k b_{n_{i+1}}). \) Then

\[ f(r_{k+1} a_{m_{i+1}}) - f(r_k a_{m_i}) \leq f(r_{k+1} a_{m_{i+1}}) \leq g(s_k b_{n_{i+1}}) \leq g(s_{k+1} b_{n_{i+1}}) - g(s_k b_{n_{i+1}}) \]

Therefore (1) is a regular basis for \( E \times F. \)
(2) \( f \succ g : (S_p(b, \infty), S_f(a, 1)) \in (S^r) \Rightarrow (f^{-1}g(b), a) \in HP \setminus 1 \).

From Lemma 2 (b)

\[ \exists n_0, k \quad \forall K, m \quad \exists n, i_0, j_0 \quad \forall i \geq i_0, \forall j \geq j_0 : \]

\[ r_k < \frac{f^{-1}g(b_j)}{a_i} \frac{f^{-1}g((s_{m} - s_{n_0})b_j)}{f^{-1}g(b_j)} \Rightarrow r_K \leq \frac{f^{-1}g(b_j)}{a_i} \frac{f^{-1}g(2s_n b_j)}{f^{-1}g(b_j)} \]

Let \( A \) be a finite limit point of \( \{ \frac{f^{-1}g(b_j)}{a_i} : i, j \in \mathbb{N} \} \). Since \( f \succ g \),

\[ \exists k \quad \forall K \quad r_k \leq A \Rightarrow r_K \leq A \]

but \( r_k \nrightarrow 1 \), so there exists \( c \in [0, 1) \), such that \( A \leq c \) or \( A \geq 1 \) or equivalently

\[ LIM \{ \frac{f^{-1}g(b_j)}{a_i} : i, j \in \mathbb{N} \} \subset [0, c] \cup [1, \infty) \]

i.e. \( (f^{-1}g(b), a) \in HP \setminus 1 \). This condition is equivalent to the following:

\[ \exists (m_i), (n_i), \text{ strictly increasing sequences, such that} \]

\( \limsup_{i \to \infty} \frac{f^{-1}g(b_{n_i+1})}{a_{m_i+1}} < 1 \) and \( \liminf_{i \to \infty} \frac{f^{-1}g(b_{n_i+1})}{a_{m_i}} \geq 1 \)

If \((x_i), (y_i)\) denote the canonical bases in \( E, F \) respectively, we claim that

\[ \ldots, x_{m_i-1+1}, \ldots, x_{m_i}, y_{n_i+1}, \ldots, y_{n_i+1}, x_{m_i+1}, \ldots \quad (2) \]

is a regular basis for \( E \times F \).

First we show that

\[ f(r_{k+1}a_{m_i}) - f(r_k a_{m_i}) \leq g(s_{k+1}b_{n_i+1}) - g(s_k b_{n_i+1}) \]

We use (ii) and convexity of \( g \)

\[ \exists i_o \quad \forall i \geq i_o \text{ and } \forall k \quad \frac{f^{-1}g(b_{n_i+1})}{a_{m_i}} \geq r_{k+1} \]

Hence

\[ f(r_{k+1}a_{m_i}) - f(r_k a_{m_i}) \leq f(r_{k+1}a_{m_i}) \]

\[ \leq g(b_{n_i+1}) \]

\[ \leq g(s_{k} b_{n_i+1}) \]

\[ \leq g(s_{k+1}b_{n_i+1}) - g(s_k b_{n_i+1}) \]

Then we show

\[ g(s_{k+1}b_{n_i+1}) - g(s_k b_{n_i+1}) \leq f(r_{k+1}a_{m_i+1}) - f(r_k a_{m_i+1}) \]
Let \( \epsilon_k = \frac{r_{k+1}}{r_{k+1} - r_k} > 1 \). Use (i) and \( f > g \):

\[ \exists k_0 \forall k \geq k_0 \exists i_o \forall i \geq i_o \frac{f^{-1}g(b_{n+i})}{a_{m+i+1}} \frac{f^{-1}g(\epsilon_k b_{n+i})}{f^{-1}g(b_{n+i})} < r_{k+1} \]

Now convexity of \( f \) and \( g \) gives

\[ g(s_{k+1}b_{n+i+1}) - g(s_kb_{n+i}) \leq g(s_{k+1}b_{n+i}) \frac{r_{k+1} - r_k}{r_{k+1}} f(r_{k+1}a_{m+i+1}) \leq f(r_{k+1}a_{m+i+1}) - f(r_k a_{m+i+1}) \]

Therefore (2) is a regular basis for \( E \times F \).

(3) \( f \approx g \) : \((S_f(a,1), S_g(b,\infty)) \in (S^*) \Rightarrow (a, b) \in HP \).

Since \( f \approx g \), there is \( \lambda > 0 \) such that \( g^{-1}f(x) = \lambda x \).

\[ \exists n_o, k \forall K, m \exists n, i_o, j_o \forall i \geq i_o, \forall j \geq j_o : \frac{s_k}{\lambda(r_{m} - r_{n_o})} < \frac{a_i}{b_j} \Rightarrow \frac{s_K}{\lambda 2r_n} \leq \frac{a_i}{b_j} \]

If \( A \) is a finite limit point of \( \{ \frac{a_i}{b_j} : i, j \in \mathbb{N} \} \), then

\[ \exists n_o, k \forall K, m \exists n \text{ such that } \frac{s_k}{\lambda(r_{m} - r_{n_o})} \leq A \Rightarrow \frac{s_K}{\lambda 2r_n} \leq A \]

which contradicts the finiteness of \( A \). Hence \( (a, b) \in HP \). Thus we have

\[ \exists (m_i), (n_i), \text{ strictly increasing sequences, such that } \]

\( (i) \sup_i \frac{a_{m+i}}{b_{n+i+1}} < \infty \) and \( (ii) \lim_{i \to \infty} \frac{a_{m+i+1}}{b_{n_i}} = \infty \)

If \((x_i), (y_i)\) denote the canonical bases in \( E, F \) respectively, we claim that (1) is a basis for \( E \times F \).

By (ii),

\[ \forall k \exists i_o \forall i \geq i_0 \frac{s_{k+1}}{\lambda(r_{k+1} - r_k)} \leq \frac{a_{m+i+1}}{b_{n_i}} \text{, since } s_{k+1}, \frac{1}{r_{k+1} - r_k} \to \infty \]

So

\[ g(s_{k+1}b_{n_i}) - g(s_kb_{n_i}) \leq g(s_{k+1}b_{n_i}) \leq g((r_{k+1} - r_k)\lambda a_{m+i+1}) \leq g(r_{k+1}\lambda a_{m+i+1}) - g(r_k \lambda a_{m+i+1}) = f(r_{k+1}a_{m+i+1}) - f(r_ka_{m+i+1}) \]
And by (i),
\[ \exists i_0, k_0 \quad \forall i \geq i_0 \forall k \geq k_0 \quad \frac{a_{m_{i+1}}}{b_{n_{i+1}}} < \frac{s_{k+1}}{2\lambda r_{k+1}} \]
Hence
\[ f(r_{k+1}a_{m_{i+1}}) - f(r_k a_{m_{i+1}}) = g(r_{k+1}a_{m_{i+1}}) - g(r_k a_{m_{i+1}}) \leq g(r_{k+1}a_{m_{i+1}}) \leq \frac{1}{2} g(s_{k+1}b_{n_{i+1}}) \leq g(s_{k+1}b_{n_{i+1}}) - g(s_k b_{n_{i+1}}) \]
Therefore (1) is a regular basis for \( E \times F \).

3.1.3 \( E = S_f(a, 1) \quad F = S_g(b, 1) \)

Lemma 3 :
(a) If \((S_f(a, 1), S_g(b, 1)) \in (S^*)\), then
\[ \exists n_0, k \quad \forall K, m \quad \exists n, i_o, j_o \quad \forall i \geq i_o, \forall j \geq j_o : \]
\[ g(s_k b_j) < f((r_m - r_{n_o})a_i) \Rightarrow g(s_K b_j) \leq f(2r_n a_i) . \]
(b) If \((S_g(b, 1), S_f(a, 1)) \in (S^*)\), then
\[ \exists n_0, k \quad \forall K, m \quad \exists n, i_o, j_o \quad \forall i \geq i_o, \forall j \geq j_o : \]
\[ f(r_k a_i) < g((s_m - s_{n_o})b_j) \Rightarrow f(r_K a_i) \leq g(2s_n b_j) . \]
The proof is exactly the same as the proof of Lemma 2, so we omit it.

Proof of the Theorem :

(1) \( f \prec g : (S_f(a, 1), S_g(b, 1)) \in (S^*) \Rightarrow (g^{-1}f(a), b) \in HP \not\prec 1 \). Lemma 3 (a) gives
\[ \exists n_0, k \quad \forall K, m \quad \exists n, i_o, j_o \quad \forall i \geq i_o, \forall j \geq j_o : \]
\[ g(s_k b_j) < f((r_m - r_{n_o})a_i) \Rightarrow g(s_K b_j) \leq f(2r_n a_i) \]
So
\[ \frac{s_k g^{-1}f(a_i)}{g^{-1}f((r_m - r_{n_o})a_i) b_j} < \frac{g^{-1}f(a_i)}{b_j} \Rightarrow s_K \frac{g^{-1}f(a_i)}{g^{-1}f(2r_n a_i) b_j} \leq \frac{g^{-1}f(a_i)}{b_j} \]
Let $A$ be a finite limit point of $\left\{ \frac{g^{-1}(a_i)}{b_j} : i, j \in \mathbb{N} \right\}$. Since $f \prec g$ we have

$$\exists k \quad \forall K \quad s_k \leq A \Rightarrow s_K \leq A$$

But $s_K \not\rightarrow 1$, so there exists $c \in [0, 1)$, such that $A \leq c$ or $A \geq 1$. This means that

$$LIM\left\{ \frac{g^{-1}(a_i)}{b_j} : i, j \in \mathbb{N} \right\} \subset [0, c] \cup [1, \infty)$$

i.e. $(g^{-1}(a), b) \in HP \land 1$. This condition is equivalent to the following:

$$\exists (m_i), (n_i),$$

strictly increasing sequences, such that

$$(i) \quad \limsup_{i \to \infty} \frac{g^{-1}(a_{m_{i+1}})}{b_{n_{i+1}}} < 1 \quad \text{and} \quad (ii) \quad \liminf_{i \to \infty} \frac{g^{-1}(a_{m_{i+1}})}{b_{n_i}} \geq 1$$

If $(x_i), (y_i)$ denote the canonical bases in $E, F$, respectively, we claim that

$$\cdots, y_{n_i-1}, \cdots, y_{n_i}, x_{m_i+1}, \cdots, x_{m_i+1}, y_{n_i+1}, \cdots \quad (1)$$

is a regular basis for $E \times F$.

In the first case we use $(ii)$ and $f \prec g$,

$$\exists i_0, \forall i \geq i_0 \quad \text{and} \quad \forall k, \quad \frac{g^{-1}(a_{m_{i+1}})}{b_{n_i}} \geq \frac{g^{-1}(a_{m_{i+1}})}{b_{n_i}} \geq s_{k+1}$$

So we get

$$g(s_{k+1}b_{n_i}) - g(s_kb_{n_i}) \leq g(s_{k+1}b_{n_i})$$

$$\leq f((r_{k+1} - r_k)a_{m_{i+1}})$$

$$\leq f(r_{k+1}a_{m_{i+1}}) - f(r_ka_{m_{i+1}})$$

In the second case put $\epsilon_k = \frac{s_{k+1}}{s_{k+1} - s_k} > 1$. Then

$$\exists k_0, \exists i_0, \forall i \geq i_0, \forall k \geq k_0, \quad \frac{g^{-1}(a_{m_{i+1}})}{b_{n_{i+1}}} \frac{g^{-1}(\epsilon_kr_{k+1}a_{m_{i+1}})}{g^{-1}(a_{m_{i+1}})} < s_{k+1}$$

by using $(i)$ and $f \prec g$. From the convexity of $f$ and $g$, we have

$$\epsilon_kf(r_{k+1}a_{m_{i+1}}) \leq f(\epsilon_kr_{k+1}a_{m_{i+1}}) \leq g(s_{k+1}b_{n_{i+1}})$$

and

$$f(r_{k+1}a_{m_{i+1}}) - f(r_ka_{m_{i+1}}) \leq f(r_{k+1}a_{m_{i+1}})$$

$$\leq \frac{s_{k+1} - s_k}{s_{k+1}} g(s_{k+1}b_{n_{i+1}})$$

$$\leq g(s_{k+1}b_{n_{i+1}}) - g(s_kb_{n_{i+1}})$$

Therefore $(1)$ is a regular basis for $E \times F$.

$$\text{(2) } f \approx g : \text{We may take } r_k = s_k \not\rightarrow 1 \text{ and follow proof of (2) in the subsection 3.1.1}$$
3.2  \((S_f(a, r), S_g(b, s)) \quad (-\infty < r, s \leq 0)\)

3.2.1  \(E = S_f(a, 0) \quad F = S_g(b, 0)\)

Lemma 4:

(a) If \((S_f(a, 0), S_g(b, 0)) \in (S^*)\), then

\[
\forall \mu \exists n_o, k \quad \forall m \exists i_o, j_o \quad \forall i \geq i_o, \forall j \geq j_o : \quad g(|s_\mu|b_j) < f(|r_{n_o}|a_i) \Rightarrow g(|s_k|b_j) \leq f(|r_m|a_i).
\]

(b) If \((S_g(b, 0), S_f(a, 0)) \in (S^*)\), then

\[
\forall \mu \exists n_o, k \quad \forall m \exists i_o, j_o \quad \forall i \geq i_o, \forall j \geq j_o : \quad f(|r_\mu|a_i) < g(|s_{n_o}|b_j) \Rightarrow f(|r_k|a_i) \leq g(|s_m|b_j).
\]

Proof: We only give the proof of (a), the other case is symmetrical.

First observe that \(r_k \not\sim 0 \Rightarrow |r_k| \searrow 0\) and \(s_k \not\sim 0 \Rightarrow |s_k| \searrow 0\). So we may choose \(2|r_{k+1}| \leq |r_k|, 2|s_{k+1}| \leq |s_k|\).

\((S_f(a, 0), S_g(b, 0)) \in (S^*)\) implies

\[
\forall \mu \exists n_o, k \quad \forall K, m \exists n, S > 0 \quad \forall i, j : \quad g(|s_k|b_j) - g(|s_K|b_j) \leq S + f(|r_m|a_i) - f(|r_n|a_i)
\]

and/or

\[
f(|r_{n_o}|a_i) - f(|r_m|a_i) \leq S + g(|s_\mu|b_j) - g(|s_k|b_j)
\]

We can find \(i_o, j_o \in \mathbb{N}\) such that for all \(i \geq i_o\), for all \(j \geq j_o\):

\[
S + f(|r_m|a_i) - f(|r_n|a_i) \leq f(|r_m|a_i)
\]

and

\[
S + g(|s_\mu|b_j) - g(|s_k|b_j) \leq g(|s_m|b_j)
\]

Since \(f\) and \(g\) are convex, for all \(K > k\) and for all \(m > n_o\) we have

\[
\frac{1}{2} \leq \frac{|s_k| - |s_K|}{|s_k|} \leq \frac{g(|s_k|b_j) - g(|s_K|b_j)}{g(|s_k|b_j)}
\]

and

\[
\frac{1}{2} \leq \frac{|r_{n_o}| - |r_m|}{|r_{n_o}|} \leq \frac{f(|r_{n_o}|a_i) - f(|r_m|a_i)}{f(|r_{n_o}|a_i)}
\]
By the assumption about \(|r_k|\) and \(|s_k|\), we have for all \(i,j\):

\[
2f(|r_m|a_i) \leq f(|r_{m-1}|a_i) \quad \text{and} \quad 2g(|s_\mu|b_j) \leq g(|s_{\mu-1}|b_j)
\]

So we get

\[
\forall \mu \ \exists n_\mu, k \quad \forall m \ \exists i_\mu, j_\mu \quad \forall i \geq i_\mu, j \geq j_\mu:

g(|s_\mu|b_j) < f(|r_{n_\mu}|a_i) \Rightarrow g(|s_k|b_j) \leq f(|r_m|a_i)
\]

**Proof of the Theorem:**

(1) \(f \succ g\) : \((S_f(a,0), S_g(b,0)) \in (S^*) \Rightarrow (a, f^{-1}g(b)) \in HP.

By using the Lemma 4 (a)

\[
\forall \mu \ \exists n_\mu, k \quad \forall m \ \exists i_\mu, j_\mu \quad \forall i \geq i_\mu, j \geq j_\mu:

g(|s_\mu|b_j) < f(|r_{n_\mu}|a_i) \Rightarrow g(|s_k|b_j) \leq f(|r_m|a_i)
\]

So

\[
\frac{1}{|r_{n_\mu}|} f^{-1}g(|s_\mu|b_j) < \frac{a_i}{f^{-1}g(b_j)} \Rightarrow \frac{1}{|r_m|} f^{-1}g(|s_k|b_j) \leq \frac{a_i}{f^{-1}g(b_j)}
\]

Since \(f \succ g\),

\[
\lim_{j \to \infty} \frac{f^{-1}g(|s_k|b_j)}{f^{-1}g(b_j)} = 1 \quad \forall k. \text{ Let } A \text{ be a finite limit point of }\{\frac{a_i}{f^{-1}g(b_j)} : i, j \in \mathbb{N}\}, \text{ then }
\]

\[
\exists n_\mu \quad \forall m \quad \frac{1}{|r_{n_\mu}|} \leq A \Rightarrow \frac{1}{|r_m|} \leq A
\]

which is a contradiction, since \(\frac{1}{|r_m|} \not\to \infty\). Hence \(LIM\{\frac{a_i}{f^{-1}g(b_j)} : i, j \in \mathbb{N}\}\) is bounded, i.e. \((a, f^{-1}g(b)) \in HP\), which is equivalent to

\[
\exists (m_i), (n_i), \text{ strictly increasing sequences, such that } (i) \sup_i \frac{a_{m_i+1}}{f^{-1}g(b_{n_i+1})} < \infty \quad \text{and} \quad (ii) \lim_{i \to \infty} \frac{a_{m_i+1}}{f^{-1}g(b_{n_i})} = \infty
\]

If \((x_i), (y_i)\) denote the canonical bases in \(E, F\) respectively, we claim that

\[
\ldots, y_{n_i+1}, \ldots, y_{n_i}, x_{m_i+1}, \ldots, x_{m_i+1}, y_{n_i+1}, \ldots
\]

(1)
is a regular basis for $E \times F$.

Since \((ii)\) holds, we have

$$\forall k \exists i_o \forall i \geq i_o \frac{1}{|r_{k+1}|} \leq \frac{a_{m+i+1}}{f^{-1}g(b_{n_i})} \Rightarrow g(|s_k|b_{n_i}) \leq f(|r_{k+1}|a_{m+i+1})$$

So

$$g(|s_k|b_{n_i}) - g(|s_{k+1}|b_{n_i}) \leq g(|s_k|b_{n_i})$$
$$\leq f(|r_{k+1}|a_{m+i+1})$$
$$\leq f(|r_k|a_{m+i+1}) - f(|r_{k+1}|a_{m+i+1})$$

In the second case we have by \((i)\) and $f \geq g$, so

$$\exists k \geq k_0 \exists i_o \forall i \geq i_o \frac{a_{m+i+1}}{f^{-1}g(b_{n_i})} \leq \frac{1}{r_k}$$

This shows that $f(|r_k|a_{m+i+1}) \leq g(|s_{k+1}|b_{n_i+1})$.

$$f(|r_k|a_{m+i+1}) - f(|r_{k+1}|a_{m+i+1}) \leq f(|r_k|a_{m+i+1})$$
$$\leq g(|s_{k+1}|b_{n_i+1})$$
$$\leq g(|s_k|b_{n_i+1}) - g(|s_{k+1}|b_{n_i+1})$$

Therefore (1) is a regular basis for $E \times F$.

\(2\) $f \approx g : \; (S_f(a, 0), S_g(b, 0)) \in (S^*) \Rightarrow (a, b) \in HP$.

We do not need this condition in showing the regularity of the basis (1), but we give the proof for completeness.

From the lemma,

$$\forall \mu \exists n_o, k \; \forall m \exists i_o, j_o \; \forall i \geq i_o, \forall j \geq j_o$$

$$\frac{1}{|r_{n_o}|} \frac{f^{-1}g(|s_\mu|b_j)}{f^{-1}g(b_j)} \leq \frac{a_i}{f^{-1}g(b_j)} \Rightarrow \frac{1}{|r_m|} \frac{f^{-1}g(|s_k|b_j)}{f^{-1}g(b_j)} \leq \frac{a_i}{f^{-1}g(b_j)}$$

So $f \approx g$ gives, there exists $\lambda > 0$ such that $f(x) = g(\lambda x)$. 

$$\forall \mu \exists n_o, k \; \forall m \exists i_o, j_o \; \forall i \geq i_o, \forall j \geq j_o$$

$$\frac{|s_\mu|}{\lambda|r_{n_o}|} < \frac{a_i}{b_j} \Rightarrow \frac{|s_k|}{\lambda|r_m|} < \frac{a_i}{b_j}$$

Let $A$ be a finite limit point of \(\{\frac{a_i}{b_j} : i, j \in \mathbb{N}\} \) then

$$\forall \mu \exists n_o, k \; \forall m \frac{|s_\mu|}{\lambda|r_{n_o}|} \leq A \Rightarrow \frac{|s_k|}{\lambda|r_m|} \leq A$$

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which is a contradiction. Hence $LIM\{\alpha_i/\beta_j : i, j \in \mathbb{N}\}$ is bounded, i.e. $(a, b) \in HP$.

Now we may take $|r_k| = |s_k| \to 0$ and find increasing sequences $(m_i)$, $(n_i)$ which satisfy

$$
\ldots \leq b_{n_i} \leq \lambda a_{m_i+1} \leq \ldots \leq \lambda a_{m_i+1} \leq b_{n_i+1} \leq \ldots
$$

Then we show, also in this case, that (1) is a regular basis for $E \times F$.

First observe that $|s_k|b_{n_i} \leq |r_k|\lambda a_{m_i+1}$, $\forall k$. Since $g$ is convex, we have

$$
g(|s_k|b_{n_i}) - g(|s_{k+1}|b_{n_i}) \leq \frac{g(|s_k|b_{n_i}) - g(|s_{k+1}|b_{n_i})}{\lambda a_{m_i+1}}(|r_k| - |r_{k+1}|)
$$

$$
\leq g(|r_k|\lambda a_{m_i+1}) - g(|r_{k+1}|\lambda a_{m_i+1})
$$

$$
= f(|r_k|a_{m_i+1}) - f(|r_{k+1}|a_{m_i+1})
$$

Similarly

$$
f(|r_k|a_{m_i+1}) - f(|r_{k+1}|a_{m_i+1}) = g(|r_k|\lambda a_{m_i+1}) - g(|r_{k+1}|\lambda a_{m_i+1})
$$

$$
\leq \frac{g(|s_k|b_{n_i+1}) - g(|s_{k+1}|b_{n_i+1})}{\lambda a_{m_i+1}}(|r_k| - |r_{k+1}|)
$$

$$
\leq g(|s_k|b_{n_i+1}) - g(|s_{k+1}|b_{n_i+1})
$$

Therefore (1) is a regular basis for $E \times F$.

### 3.2.2

$E = S_f(a, 0)$  \hspace{1cm} $F = S_g(b, -1)$

**Lemma 5** :

(a) If $(S_f(a, 0), S_g(b, -1)) \in (S^*)$, then

$$
\forall \mu \ \exists n_0, k \ \forall K, m \ \exists n_i, i_0, j_0 \ \forall i \geq i_0, \forall j \geq j_0 : 
$$

$$
f(|r_m|a_i) < g(|s_k| - |s_K|)b_j \ \Rightarrow \ f(|r_{n_0}|a_i) \leq g(2|s_m|b_j).
$$

(b) If $(S_g(b, -1), S_f(a, 0)) \in (S^*)$, then

$$
\forall \mu \ \exists n_0, k \ \forall K, m \ \exists n_i, i_0, j_0 \ \forall i \geq i_0, \forall j \geq j_0 : 
$$

$$
g(|s_m|b_j) < f(|r_k| - |r_K|)a_i \ \Rightarrow \ g(|s_{n_0}|b_j) \leq f(2|r_m|a_i).
$$
Proof :

(a) \((S_f(a, 0), S_g(b, -1)) \in (S^*)\) implies

\[
\forall \mu \ \exists n_o, k \ \forall K, m \ \exists n, S > 0 \ \forall i, j
\]

\[
f(|r_{n_0}|a_i) - f(|r_m|a_i) \leq S + g(|s_\mu|b_j) - g(|s_k|b_j)
\]

and/or

\[
g(|s_k|b_j) - g(|s_K|b_j) \leq S + f(|r_m|a_i) - f(|r_n|a_i)
\]

Then

\[
\forall \mu \ \exists n_o, k \ \forall K, m \ \exists n, i_o, j_o \ \forall i \geq i_o, \forall j \geq j_o
\]

\[
f(|r_{n_0}|a_i) - f(|r_m|a_i) \leq g(|s_\mu|b_j)
\]

and/or

\[
g(|s_k|b_j) - g(|s_K|b_j) \leq f(|r_m|a_i)
\]

or equivalently

\[
\forall \mu \ \exists n_o, k \ \forall K, m \ \exists n, i_o, j_o \ \forall i \geq i_o, \forall j \geq j_o
\]

\[
f(|r_m|a_i) < g((|s_k| - |s_K|)b_j) \ \Rightarrow \ f(|r_{n_0}|a_i) \leq f(|r_m|a_i) + g(|s_\mu|b_j)
\]

So

\[
\forall \mu \ \exists n_o, k \ \forall K, m \ \exists n, i_o, j_o \ \forall i \geq i_o, \forall j \geq j_o
\]

\[
f(|r_m|a_i) < g((|s_k| - |s_K|)b_j) \ \Rightarrow \ f(|r_{n_0}|a_i) \leq g(2|s_\mu|b_j)
\]

Proof of the Theorem :

(1) \(f \succ g:\ (S_f(a, 0), S_g(b, -1)) \in (S^*) \ \Rightarrow \ (a, f^{-1}(g(b))) \in HP\).

This can be proven in an exactly similar way as we have done in (1) of the proof of theorem in the previous section 3.2.1. Then similarly we show (1) is a regular basis for \(E \times F\).

In the first case put \(c_k = \frac{|r_k|}{|r_k| - |r_{k+1}|} > 1\). Then

\[
\exists i_o \ \forall i \geq i_o \text{ and } \forall k, \ \frac{a_{m_i+1}}{f^{-1}g(b_{n_i})} \frac{f^{-1}g(b_{n_i})}{f^{-1}g(c_k|s_k|b_{n_i})} \geq \frac{1}{|r_k|}
\]

by using (ii) and \(f \succ g\). From the convexity of \(f\) and \(g\), we have

\[
c_kg(|s_k|b_{n_i}) \leq g(c_k|s_k|b_{n_i}) \leq f(|r_k|a_{m_i+1}).
\]
\[ g(|s_k|b_{n_i}) - g(|s_{k+1}|b_{n_i}) \leq g(|s_k|b_{n_i}) \]
\[ \leq \frac{|r_k| - |r_{k+1}|}{|r_k|} f(|r_k|a_{m_i+1}) \]
\[ \leq f(|r_k|a_{m_i+1}) - f(|r_{k+1}|a_{m_i+1}) \]

In the second case we use (i) and \( f > g \),
\[ \exists i_o, k_o \forall i \geq i_o \text{ and } \forall k \geq k_o \quad \frac{a_{m_i+1}}{f^{-1}g(b_{n_i+1})} \leq \frac{f^{-1}g(b_{n_i+1})}{f^{-1}g(|s_k| - |s_{k+1}|b_{n_i+1})} \leq \frac{1}{|r_k|} \]

So we get
\[ f(|r_k|a_{m_i+1}) - f(|r_{k+1}|a_{m_i+1}) \leq f(|r_k|a_{m_i+1}) \]
\[ \leq g((|s_k| - |s_{k+1}|b_{n_i+1}) \]
\[ \leq g(|s_k|b_{n_i+1}) - g(|s_{k+1}|b_{n_i+1}) \]

(2) \( f \prec g \) : Using Lemma 5 (b) we may prove :
\[ (S_g(b, -1), S_f(a, 0)) \in (S^*) \quad \Rightarrow \quad (b, g^{-1}f(a)) \in HP \nabla 1 \]
in a similar way in (2) of the proof of the theorem in 3.1.2.

This condition is equivalent to the below one :

\[ \exists (m_i), (n_i), \text{ strictly increasing sequences, such that} \]

\[ \text{(i)} \quad \limsup_{i \to \infty} \frac{b_{n_i+1}}{g^{-1}f(a_{m_i+1})} < 1 \quad \text{and} \quad \text{(ii)} \quad \liminf_{i \to \infty} \frac{b_{n_i+1}}{g^{-1}f(a_{m_i})} \geq 1 \]

If \((x_i), (y_i)\) denote the canonical bases in \( E, F \) respectively, we claim that
\[ \ldots, x_{m_{i-1}+1}, \ldots, x_{m_i}, y_{n_{i+1}+1}, \ldots, y_{n_{i+1}}, x_{m_{i+1}}, \ldots \quad (2) \]
is a regular basis for \( E \times F \).

Use (ii) to get
\[ \forall k \exists i_o \forall i \geq i_o \quad \frac{b_{n_i+1}}{g^{-1}f(a_{m_i})} \geq \frac{1}{|s_k|} \]

Then since \(|r_k| \leq 0\), \( \exists k_o \forall k \geq k_o, \quad |r_k| < 1. \)
\[ f(|r_k|a_{m_i}) - f(|r_{k+1}|a_{m_i}) \leq f(|r_k|a_{m_i}) \]
\[ \leq f(a_{m_i}) \]
\[ \leq g(|s_{k+1}|b_{n_i+1}) \]
\[ \leq g(|s_k|b_{n_i+1}) - g(|s_{k+1}|b_{n_i+1}) \]
In the second case we use (i) and \( f \sim g \)

\[
\exists k_0 \quad \forall k \geq k_0 \exists i_0 \quad \forall i \geq i_0 \quad \frac{b_{n_{i+1}}}{g^{-1}f(a_{m_{i+1}})} \cdot \frac{g^{-1}f(a_{m_{i+1}})}{g^{-1}f(a_{m_{i+1}})} \leq \frac{1}{|s_k|}
\]

Then

\[
g(|s_k|b_{n_{i+1}}) - g(|s_{k+1}|b_{n_{i+1}}) \leq g(|s_k|b_{n_{i+1}})
\]

\[
\leq f(|r_{k+1}|a_{m_{i+1}})
\]

\[
\leq f(|r_{k+1}|a_{m_{i+1}}) - f(|r_{k+1}|a_{m_{i+1}})
\]

Therefore (2) is a regular basis for \( E \times F \).

(3) \( f \approx g : (S_f(a,0), S_g(b,-1)) \in (S^*) \Rightarrow (a,b) \in HP \).

In this case we have

\[
\exists n_o, k \quad \forall m \quad A \geq \frac{2\lambda}{|r_{n_o}|} \Rightarrow A \geq \frac{\lambda(|s_k| - 1)}{|r_m|}
\]

where \( A \) is a finite limit point of \( \{\frac{a_i}{b_j}: i, j \in \mathbb{N}\} \). Hence \((a,b) \in HP \). Thus we have

\[
\exists (m_i), (n_i), \text{ strictly increasing sequences, such that}
\]

\[
(i) \quad \sup_i \frac{a_{m_{i+1}}}{b_{n_{i+1}}} < \infty \quad \text{and} \quad (ii) \quad \lim_{i \to \infty} \frac{a_{m_{i+1}}}{b_{n_i}} = \infty
\]

If \((x_i), (y_i)\) denote the canonical bases in \( E, F \) respectively, we claim that

(1) is a basis for \( E \times F \).

We assume \( 2|r_{k+1}| \leq |r_k| \) since \( |r_k| \searrow 0 \) and put \( |r_k| = |s_k| - |s_{k+1}| \), where \( |s_k| \searrow 1 \). By (ii),

\[
\forall k \exists i_o \quad \forall i \geq i_o \quad \frac{|s_k|}{\lambda|r_{k+1}|} \leq \frac{a_{m_{i+1}}}{b_{n_i}} \quad \text{since} \quad |s_k| \searrow 1, \quad \frac{1}{|r_k|} \nearrow \infty
\]

So

\[
g(|s_k|b_{n_i}) - g(|s_{k+1}|b_{n_i}) \leq g(|s_k|b_{n_i})
\]

\[
\leq g(|r_{k+1}|la_{m_{i+1}})
\]

\[
\leq g(|r_k|\lambda a_{m_{i+1}}) - g(|r_{k+1}|\lambda a_{m_{i+1}})
\]

\[
= f(|r_k|a_{m_{i+1}}) - f(|r_{k+1}|a_{m_{i+1}})
\]

Now we use (i), \( f \approx g \) and the equivalence relation between \( |r_k| \) and \( |s_k| \). Moreover since \( S_f(a,0) \equiv S_f(\tilde{a},0) \) where \( \tilde{a}_i = ca_i \) for some constant \( c > 0 \),
we may assume without loss of generality that \( \frac{\lambda a_{m+1}}{b_{n+1}} \leq 1 \). Hence

\[
f(|r_k|a_{m+1}) - f(|r_{k+1}|a_{m+1}) \leq f(|r_k|a_{m+1}) - f(|r_{k+1}|a_{m+1}) \leq g(|s_k| - |s_{k+1}|) \lambda a_{m+1}) \leq g(|s_k|b_{n+1}) - g(|s_{k+1}|b_{n+1})
\]

Therefore (1) is a regular basis for \( E \times F \).

**3.2.3** \( E = S_f(a, -1) \quad F = S_g(b, -1) \)

**Lemma 6 :**

(a) If \((S_f(a,-1), S_g(b,-1)) \in (S^*)\), then

\[
\forall \mu \exists n_0, k \quad \forall K, m \quad \exists n, i_o, j_o \quad \forall i \geq i_o, \forall j \geq j_o: \quad f(|r_m|a_i) < g((|s_k| - |s_K|)b_j) \Rightarrow f(|r_{n_0}|a_i) \leq g(2|s_{i_0}|b_j).
\]

(b) If \((S_g(b,-1), S_f(a,-1)) \in (S^*)\), then

\[
\forall \mu \exists n_0, k \quad \forall K, m \quad \exists n, i_o, j_o \quad \forall i \geq i_o, \forall j \geq j_o: \quad g(|s_m|b_j) < f(|r_k| - |r_K|)a_i) \Rightarrow g(|s_{n_0}|b_j) \leq f(2|r_m|a_i).
\]

**Proof :** It is the same as the proof of Lemma 5.

**Proof of the Theorem :**

(1) \( f \succ g : (S_f(a,-1), S_g(b,-1)) \in (S^*) \Rightarrow (a, f^{-1}g(b)) \in HP \chi 1 \).

By Lemma 6 (a)

\[
\forall \mu \exists n_0, k \quad \forall K, m \quad \exists i_o, j_o \quad \forall i \geq i_o, \forall j \geq j_o: \quad f(|r_m|a_i) < g((|s_k| - |s_K|)b_j) \Rightarrow f(|r_{n_0}|a_i) \leq g(2|s_{i_0}|b_j)
\]

So

\[
\frac{a_i}{f^{-1}g(b_j)} < \frac{1}{|r_m|} \frac{f^{-1}g(|s_k| - |s_K|)b_j)}{f^{-1}g(b_j)} \Rightarrow \frac{a_i}{f^{-1}g(b_j)} \leq \frac{1}{|r_{n_0}|} \frac{f^{-1}g(2|s_{i_0}|b_j)}{f^{-1}g(b_j)}
\]

Let \( A \) be a finite limit point of \( \{ \frac{a_i}{f^{-1}g(b_j)} : i, j \in \mathbb{N} \} \). Since \( f \succ g \),

\[
\exists n_o \quad \forall m \quad \frac{1}{|r_{n_0}|} \leq A \Rightarrow \frac{1}{|r_m|} \leq A.
\]
But $\frac{1}{|r_m|} \not\leq 1$, so $(a, f^{-1}g(b)) \in HP \chi^1$. This condition is equivalent to the below one:

\[\exists (m_i), (n_i), \text{strictly increasing sequences, such that}\]

\[(i) \quad \limsup_{i \to \infty} \frac{a_{m_{i+1}}}{f^{-1}g(b_{n_{i+1}})} < 1 \quad \text{and} \quad (ii) \quad \liminf_{i \to \infty} \frac{a_{m_{i+1}}}{f^{-1}g(b_{n_i})} \geq 1\]

If $(x_i), (y_i)$ denote the canonical bases in $E, F$ respectively, we claim that

$$
\cdots, y_{n_{i-1}+1}, \cdots, y_{n_i}, x_{m_i+1}, \cdots, x_{m_i+1}, y_{n_i+1}, \cdots \quad (1)
$$

is a regular basis for $E \times F$.

But this can be proven in exactly same way as given in the proof of (1) in the section 3.2.2.

(2) $f \prec g : (S_g(b, -1), S_f(a, -1)) \in (S^*) \Rightarrow (b, g^{-1}f(a)) \in HP \chi^1$. Similar to (2) of the theorem in 3.2.2.

(3) $f \approx g :$ We may put $|r_k| = |s_k| \not\leq 1$ and apply the same proof as we have done in the subsection 3.1.1.

### 3.3

$$(S_f(a, r), S_g(b, s)) \quad (-\infty < r \leq 0, 0 < s \leq \infty)$$

### 3.3.1

$E = S_f(a, 0) \quad F = S_g(b, \infty)$

**Lemma 7 :**

If $(S_f(a, 0), S_g(b, \infty)) \in (S^*)$, then

(a) $\exists n_o, k \quad \forall K, m \quad \exists i_o, j_o \quad \forall i \geq i_o, \forall j \geq j_o :$

\[f(|r_m|a_i) < g(s_K b_j) \quad \Rightarrow \quad f(|r_{n_o}|a_i) \leq g(s_K b_j)\]

(b) $\exists n_o, k \quad \forall K, m \quad \exists i_o, j_o \quad \forall i \geq i_o, \forall j \geq j_o :$

\[g(s_K b_j) < f(|r_{n_o} - |r_m|a_i) \quad \Rightarrow \quad g(s_K b_j) \leq f(|r_{n_o}|a_i)\]

**Proof :** Observe that $r_k \not\leq 0 \Rightarrow |r_k| \not\leq 0$ and $s_k \not\leq \infty$. So we may choose $2|r_{k+1}| \leq |r_k|, 2s_k \leq s_{k+1}$. $(S_f(a, 0), S_g(b, \infty)) \in (S^*)$ implies
\[
\forall \mu \exists n_0, k \quad \forall K, m \quad \exists n, S > 0 \quad \forall i, j : \\
g(s_K b_j) - g(s_k b_j) \leq S + f(|r_m| a_i) - f(|r_n| a_i)
\]
and/or
\[
f(|r_{n_0}| a_i) - f(|r_m| a_i) \leq S + g(s_k b_j) - g(s_k b_j)
\]
We can find \( i_o, j_o \in \mathbb{N} \) such that for all \( i \geq i_o \), all \( j \geq j_o \):
\[
S + f(|r_m| a_i) - f(|r_{n_0}| a_i) \leq f(|r_m| a_i)
\]
and
\[
S + g(s_k b_j) - g(s_k b_j) \leq g(s_k b_j)
\]
So \( \exists n_0, k \quad \forall K, m \quad \exists i_o, j_o \quad \forall i \geq i_o, \forall j \geq j_o : \\
g(s_k b_j) - g(s_k b_j) \leq f(|r_m| a_i)
\]
and/or
\[
f(|r_{n_0}| a_i) - f(|r_m| a_i) \leq g(s_k b_j) \quad (*)
\]
(a) Since \( f \) and \( g \) are convex, for all \( K > k \) and for all \( m > n_0 \) we have
\[
\frac{1}{2} \leq \frac{s_k - s_K}{s_K} \leq \frac{g(s_k b_j) - g(s_k b_j)}{g(s_k b_j)}
\]
and
\[
\frac{1}{2} \leq \frac{|r_n| - |r_m|}{|r_{n_o}|} \leq \frac{f(|r_{n_0}| a_i) - f(|r_m| a_i)}{f(|r_{n_0}| a_i)}
\]
So \( \exists n_0, k \quad \forall K, m \quad \exists i_o, j_o \quad \forall i \geq i_o, j \geq j_o : \\
\frac{1}{2} g(s_K b_j) \leq f(|r_m| a_i) \quad \text{and/or} \quad \frac{1}{2} f(|r_{n_0}| a_i) \leq g(s_k b_j)
\]
From \( 2|r_m| \leq |r_{m-1}| \) and \( 2s_k \leq s_{k+1} \) and convexity it follows that
\[
2f(|r_m| a_i) \leq f(|r_{m-1}| a_i) \quad \text{and} \quad 2g(s_k b_j) \leq g(s_{k+1} b_j)
\]
\( \exists n_0, k \quad \forall K, m \quad \exists i_o, j_o \quad \forall i \geq i_o, \forall j \geq j_o : \\
f(|r_m| a_i) < g(s_K b_j) \Rightarrow f(|r_{n_0}| a_i) \leq g(s_k b_j)
\]
(b) From (*)
\[
\exists n_0, k \quad \forall K, m \quad \exists i_o, j_o \quad \forall i \geq i_o, \forall j \geq j_o :
\]
\[
g(s_k b_j) < f(|r_{n_0} - |r_m|) a_i) \leq f(|r_{n_0}| a_i) - f(|r_m| a_i)
\]
\[
\Rightarrow g(s_K b_j) \leq f(|r_m| a_i) + g(s_k b_j) \leq f(|r_{n_0}| a_i)
\]

**Proof of the Theorem:**

(1) \(f \succ g: (S_1(a,0), S_2(b, \infty)) \in (S^*) \Rightarrow (a, f^{-1} g(b)) \in HP.

By using the Lemma 7 (a)

\[
\exists n_0, K, m \exists i_o, j_o \forall i \geq i_o, \forall j \geq j_o:
\]

\[
f(|r_m| a_i) < g(s_K b_j) \Rightarrow f(|r_{n_0}| a_i) \leq g(s_k b_j)
\]

So

\[
\frac{1}{|r_{n_0}|} f^{-1} g(s_k b_j) < \frac{a_i}{f^{-1} g(b_j)} \Rightarrow \frac{1}{|r_m|} f^{-1} g(s_K b_j) \leq \frac{a_i}{f^{-1} g(b_j)}
\]

Let \(A\) be a finite limit point of \(\{\frac{a_i}{f^{-1} g(b_j)} : i, j \in N,\}\) then

\[
\exists n_0 \forall m \frac{1}{|r_{n_0}|} \leq A \Rightarrow \frac{1}{|r_m|} \leq A
\]

which is a contradiction, since \(\frac{1}{|r_m|} \to \infty\). Hence \(LIM\{\frac{a_i}{f^{-1} g(b_j)} : i, j \in N,\}\) is bounded, i.e. \((a, f^{-1} g(b)) \in HP\), which is equivalent to

\[
\exists (m_i), (n_i), \text{strictly increasing sequences, such that}
\]

(i) \(\sup_i \frac{a_{m_{i+1}}}{f^{-1} g(b_{n_{i+1}})} < \infty\) and (ii) \(\lim_{i \to \infty} \frac{a_{m_{i+1}}}{f^{-1} g(b_{n_i})} = \infty\)

If \((x_i), (y_i)\) denote the canonical bases in \(E, F\) respectively, we claim that

\[
\ldots, y_{n_{i-1}+1}, \ldots, y_{n_i}, x_{m_i+1}, \ldots, x_{m_{i+1}}, y_{n_{i+1}}, \ldots
\]

(1)

is a regular basis for \(E \times F\).

Since (ii) holds, we have same \(i_o, k_o\) such that for all \(i \geq i_o\) for all \(k \geq k_o\)

\[
\frac{1}{|r_{k+1}|} \leq \frac{a_{m_{i+1}}}{f^{-1} g(b_{n_{i+1}})} \Rightarrow g(s_{k+1} b_{n_i}) \leq f(|r_{k+1}| a_{m_{i+1}})
\]

So

\[
g(s_{k+1} b_{n_i}) - g(s_k b_{n_i}) \leq g(s_{k+1} b_{n_i})
\]

\[
\leq f(|r_{k+1}| a_{m_{i+1}})
\]

\[
\leq f(|r_k| a_{m_{i+1}}) - f(|r_{k+1}| a_{m_{i+1}})
\]

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In the second case we have by (i) and $f > g$, so
\[ \exists k_0 \forall k \geq k_0 \exists i_0 \forall i \geq i_0, \quad \frac{a_{m_k+1}}{f^{-1}g(b_{n_k+1})} \leq \frac{1}{|r_k|} \]
This shows that $f(|r_k|a_{m_k+1}) \leq g(b_{n_k+1}) \leq g(s_k b_{n_k+1})$.
\[
f(|r_k|a_{m_k+1}) - f(|r_{k+1}|a_{m_{k+1}}) \leq f(|r_k|a_{m_k+1}) \leq g(s_k b_{n_k+1}) \leq g(s_{k+1} b_{n_k+1}) - g(s_k b_{n_k+1})
\]
Therefore (1) is a regular basis for $E \times F$.

(2) $f \prec g \ni (S_f(a, 0), S_g(b, \infty)) \in (S^*) \Rightarrow (g^{-1}f(a), b) \in HP$.
This can be proven also in a similar way, so we do not need to give the proof. Equivalently we have
\[
\exists (m_i), (n_i), \text{ strictly increasing sequences, such that}
\]
\[
(i) \quad \sup_i g^{-1}f(a_{m_i+1}) < \infty \quad \text{and} \quad (ii) \lim_{i \to \infty} g^{-1}f(a_{m_i+1}) = \infty
\]
Then we show that (1) is a regular basis for $E \times F$. Similar to the previous one, in the first step we use (ii) and $f \prec g$ to get
\[
g(s_{k+1} b_{n_i}) - g(s_k b_{n_i}) \leq f(|r_k|a_{m_k+1}) - f(|r_{k+1}|a_{m_{k+1}})
\]
And secondly use (i) to get
\[
f(|r_k|a_{m_k+1}) - f(|r_{k+1}|a_{m_{k+1}}) \leq g(s_{k+1} b_{n_i+1}) - g(s_k b_{n_i+1})
\]
Therefore (1) is a regular basis for $E \times F$.

(3) $f \sim g \ni (S_f(a, 0), S_g(b, \infty)) \in (S^*) \Rightarrow (a, b) \in HP$.
From the lemma,
\[
\exists n_o, k \quad \forall K, m \quad \exists i_o, j_o \quad \forall i \geq i_o, \forall j \geq j_o:
\]
\[
\frac{1}{|r_{n_o}|} f^{-1}g(s_k b_j) \leq \frac{a_i}{f^{-1}g(b_j)} \quad \Rightarrow \quad \frac{1}{|r_m|} f^{-1}g(s_K b_j) \leq \frac{a_i}{f^{-1}g(b_j)}
\]
So $f \sim g$ gives
\[
\exists n_o, k \quad \forall K, m \quad \exists i_o, j_o \quad \forall i \geq i_o, \forall j \geq j_o:
\]
\[
\frac{s_k}{\lambda|r_{n_o}|} < \frac{a_i}{b_j} \quad \Rightarrow \quad \frac{s_K}{\lambda|r_m|} \leq \frac{a_i}{b_j}
\]

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Let $A$ be a finite limit point of $\left\{ \frac{a_i}{b_j} : i, j \in \mathbb{N} \right\}$ then

$$\exists n_0, k \quad \forall K, m \quad \frac{s_k}{\lambda|r_{n_0}|} \leq A \quad \Rightarrow \quad \frac{s_K}{\lambda|r_m|} \leq A$$

which is a contradiction. Hence $LIM\{\frac{a_i}{b_j} : i, j \in \mathbb{N}\}$ is bounded, i.e. $(a, b) \in HP$. Thus we have

$$\exists (m_i), (n_i), \text{ strictly increasing sequences, such that}$$

$$(i) \quad \sup_{i} \frac{a_{m_i+1}}{b_{n_i+1}} < \infty \quad \text{and} \quad (ii) \quad \lim_{i \to \infty} \frac{a_{m_i+1}}{b_{n_i}} = \infty$$

If $(x_i), (y_i)$ denote the canonical bases in $E, F$ respectively, we claim that (1) is a regular basis for $E \times F$.

By (ii)

$$\forall k \exists i_0 \forall i \geq i_0 \quad \frac{s_{k+1}}{\lambda|r_{k+1}|} \leq \frac{a_{m_i+1}}{b_{n_i}} \Rightarrow s_{k+1}b_{n_i} \leq \lambda|r_{k+1}|a_{m_i+1}$$

So

$$g(s_{k+1}b_{n_i}) - g(s_kb_{n_i}) \leq g(s_{k+1}b_{n_i}) \leq g(\lambda|r_{k+1}|a_{m_i+1}) \leq f(\lambda|r_{k+1}|a_{m_i+1}) \leq f(\lambda|r_{k+1}|a_{m_i+1}) - f(\lambda|r_{k+1}|a_{m_i+1})$$

By (i)

$$\exists k_0 \forall k \geq k_0 \exists i_0 \forall i \geq i_0 \quad \frac{a_{m_i+1}}{b_{n_i+1}} \leq \frac{s_k}{\lambda|r_k|}$$

This shows that $\lambda|r_k|a_{m_i+1} \leq s_k b_{n_{i+1}}$. So

$$f(\lambda|r_k|a_{m_i+1}) - f(\lambda|r_{k+1}|a_{m_i+1}) \leq g(\lambda|r_k|a_{m_i+1}) \leq g(s_k b_{n_{i+1}}) \leq g(s_{k+1} b_{n_{i+1}}) - g(s_k b_{n_{i+1}})$$

Therefore (1) is a regular basis for $E \times F$.

3.3.2 $E = S_f(a, -1) \quad F = S_g(b, \infty)$

**Lemma 8**: If $(S_f(a, -1), S_g(b, \infty)) \in (S^*)$, then

$$\exists n_0, k \quad \forall K, m \quad \exists i_0, j_0 \quad \forall i \geq i_0, \forall j \geq j_0 :$$

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(a) \[ f(|r_m|a_i) < g((s_K - s_k)b_j) \Rightarrow f(|r_{n_o}|a_i) \leq g(s_K b_j). \]

(b) \[ g(s_k b_j) < f((|r_{n_o}| - |r_m|)a_i) \Rightarrow g(s_K b_j) \leq f(|r_{n_o}|a_i). \]

**Proof:** \((S_f(a,-1), S_g(b,\infty)) \in (S^*)\) implies

\[ \forall \mu \ \exists n_o, k \ \forall K, m \ \exists n, S > 0 \ \forall i, j : 
\]

\[ g(s_k b_j) - g(s_k b_j) \leq S + f(|r_m|a_i) - f(|r_{n_o}|a_i) \]

and/or

\[ f(|r_{n_o}|a_i) - f(|r_m|a_i) \leq S + g(s_k b_j) - g(s_k b_j) \]

If we proceed as in the cases above, we get

\[ \exists n_o, k \ \forall K, m \ \exists i_o, j_o \ \forall i \geq i_o, \forall j \geq j_o : 
\]

\[ g(s_K b_j) - g(s_k b_j) \leq f(|r_m|a_i) \]

and/or

\[ f(|r_{n_o}|a_i) - f(|r_m|a_i) \leq g(s_k b_j) \]

So, \((S_f(a,-1), S_g(b,\infty)) \in (S^*)\) implies

\[ \exists n_o, k \ \forall K, m \ \exists i_o, j_o \ \forall i \geq i_o, \forall j \geq j_o : 
\]

(1) \[ f > g : (S_f(a,-1), S_g(b,\infty)) \in (S^*) \Rightarrow (a, f^{-1}g(b)) \in HP \not\subset 1. \]

From the lemma (a) and putting \(A\) to be a finite limit point of \(\{\frac{a_i}{f^{-1}g(b_j)} : i, j \in \mathbb{N}\}\),

\[ \exists n_o \ \forall m \ A \geq \frac{1}{|r_{n_o}|} \Rightarrow A \geq \frac{1}{|r_m|} \]

which shows that \((a, f^{-1}g(b)) \in HP \not\subset 1. \) This condition is equivalent to the below one:

\[ \exists (m_i), (n_i), \text{ strictly increasing sequences, such that} \]

(i) \[ \limsup_{i \to \infty} \frac{a_{m_{i+1}}}{f^{-1}g(b_{n_{i+1}})} < 1 \] \[ \text{ and } \]

(ii) \[ \liminf_{i \to \infty} \frac{a_{m_{i+1}}}{f^{-1}g(b_{n_i})} \geq 1 \]
If \((x_i), (y_i)\) denote the canonical bases in \(E, F\) respectively, we claim that

\[
\cdots, y_{n-m+1}, \cdots, y_n, x_{m+1}, \cdots, x_{m+n+1}, y_{n+1}, \cdots
\]

is a regular basis for \(E \times F\).

In the first case we use (\(ii\)) and \(f \succ g\) and put \(\epsilon_k = \frac{|r_k|}{|r_k| - |r_{k+1}|} > 1\),

\[
\forall k \exists i_o \quad \forall i \geq i_o \quad \frac{a_{m+1}^i}{f^{-1}g(b_n^i)} \geq \frac{1}{|r_k|}
\]

So we get

\[
\epsilon_k g(s_{k+1}b_n^i) \leq g(\epsilon_k s_{k+1}b_n^i) \leq f(|r_k|a_{m+1}^i)
\]

Hence

\[
g(s_{k+1}b_n^i) - g(s_kb_n^i) \leq \frac{1}{\epsilon_k} f(|r_k|a_{m+1}^i) \leq f(|r_k|a_{m+1}^i) - f(|r_{k+1}|a_{m+1}^i)
\]

In the second case

\[
\exists i_o, k_o \quad \forall i \geq i_o \quad \exists k \geq k_o \quad a_{m+1}^i \frac{f^{-1}g(b_n^i)}{f^{-1}g(b_{n+1}^i)} \frac{f^{-1}g((s_{k+1}^i - s_k)b_{n+1}^i)}{f^{-1}g((s_{k+1}^i - s_k)b_{n+1}^i)} < \frac{1}{|r_k|}
\]

by using (\(i\)) and \(f \succ g\).

\[
f(|r_k|a_{m+1}^i) - f(|r_{k+1}|a_{m+1}^i) \leq f(|r_k|a_{m+1}^i) \leq g((s_{k+1}^i - s_k)b_{n+1}^i) \leq g(s_{k+1}^i b_{n+1}^i) - g(s_kb_{n+1}^i)
\]

Therefore (1) is a regular basis for \(E \times F\).

(2) \(f \prec g\) : \((S_f(a, -1), S_g(b, \infty)) \in (S^*) \Rightarrow (g^{-1}f(a), b) \in HP\).

From (\(a\))

\[
\exists n_o, k \quad \forall K, m \quad \exists n, i_o, j_o \quad \forall i \geq i_o, \forall j \geq j_o : \\
\frac{g^{-1}f(a_i)}{g^{-1}f(|r_{n_o}| - |r_m|a_i)} \leq \frac{g^{-1}f(a_i)}{b_j} \Rightarrow s_K \frac{g^{-1}f(a_i)}{g^{-1}f(r_{n_o}a_i)} \leq \frac{g^{-1}f(a_i)}{b_j}
\]

Let \(A\) be a finite limit point of \(\{\frac{g^{-1}f(a_i)}{b_j} : i, j \in \mathbb{N}\}\). Since \(f \prec g\),

\[
\exists k \quad \forall K \quad s_k \leq A \quad \Rightarrow \quad s_K \leq A
\]

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which is a contradiction, hence \( \lim \left\{ \frac{g^{-1}f(a_i)}{b_j} : i, j \in \mathbb{N} \right\} \) is bounded, i.e. 
\((g^{-1}f(a), b) \in HP\), which is equivalent to

\[
\exists (m_i), (n_i), \ \text{strictly increasing sequences, such that}
\]

\[
(i) \ \sup_i \frac{g^{-1}f(a_{m_i+1})}{b_{n_i+1}} < \infty \quad \text{and} \quad (ii) \ \lim_{i \to \infty} \frac{g^{-1}f(a_{m_i+1})}{b_{n_i}} = \infty
\]

If \((x_i), (y_i)\) denote the canonical bases in \(E, F\) respectively, we claim that

\[
\ldots, y_{n_i-1+1}, \ldots, y_{n_i}, x_{m_i+1}, \ldots, x_{m_i+1}, y_{n_i+1}, \ldots
\]

is a regular basis for \(E \times F\).

First from (ii) and \( f < g \),

\[
\exists i_0, k_0 \ \forall i \geq i_0 \ \text{and} \ \forall k \geq k_0, \quad \frac{g^{-1}f(a_{m_i+1})}{b_{n_i+1}} \frac{g^{-1}f(|r_k| - |r_{k+1}|a_{m_i+1})}{g^{-1}f(a_{m_i+1})} \geq s_{k+1}
\]

So we get

\[
g(s_{k+1}b_{n_i}) - g(s_kb_{n_i}) \leq g(s_{k+1}b_{n_i}) \leq g((|r_k| - |r_{k+1}|a_{m_i+1}) \leq g(|r_k|a_{m_i+1} - f(|r_{k+1}|a_{m_i+1})
\]

Next consider (i) and \( f < g \)

\[
\exists i_0, k_0 \ \forall i \geq i_0 \ \text{and} \ \forall k \geq k_0, \quad \frac{g^{-1}f(a_{m_i+1})}{b_{n_i+1}} \frac{g^{-1}f(|r_k|a_{m_i+1})}{g^{-1}f(a_{m_i+1})} < s_k
\]

which shows \( f(|r_k|a_{m_i+1}) \leq g(s_kb_{n_i+1}) \). Then

\[
f(|r_k|a_{m_i+1}) - f(|r_{k+1}|a_{m_i+1}) \leq f(|r_k|a_{m_i+1}) \leq g(s_kb_{n_i+1}) \leq g(s_{k+1}b_{n_i+1}) - g(s_kb_{n_i+1})
\]

Therefore (1) is a regular basis for \(E \times F\).

(3) \( f \approx g : (S_f(a,-1), S_g(b,\infty)) \in (S^*) \Rightarrow (a, b) \in HP\).

Since \( f \approx g \), there exists \( \lambda > 0 \) such that \( g^{-1}f(x) = \lambda x \).

\[\exists n_o, k \ \forall K, m \ \exists i, i_o, j_o \ \forall i \geq i_o, \forall j \geq j_o : \]

\[
\frac{s_k}{\lambda(|r_{n_o}| - |r_m|)} < \frac{a_i}{b_j} \Rightarrow \frac{s_K}{\lambda|r_{n_o}|} \leq \frac{a_i}{b_j}
\]

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If $A$ is a finite limit point of $\left\{ \frac{a_i}{b_j} : i, j \in \mathbb{N} \right\}$, then
\[
\exists n_o, k \quad \forall K, m \quad \exists n \quad \frac{s_k}{\lambda(|r_{n_o}| - |r_m|)} \leq A \quad \Rightarrow \quad \frac{s_K}{\lambda|r_{n_o}|} \leq A
\]
which contradicts to the finiteness of $A$. Hence $(a, b) \in HP$. Thus we have
\[
\exists \ (m_i), (n_i), \text{ strictly increasing sequences, such that}
\]
(i) $\sup_i \frac{a_{m_{i+1}}}{b_{n_{i+1}}} < \infty$ and (ii) $\lim_{i \to \infty} \frac{a_{m_{i+1}}}{b_{n_i}} = \infty$

If $(x_i)$, $(y_i)$ denote the canonical bases in $E, F$ respectively, we claim that (1) is a regular basis for $E \times F$.

By (ii),
\[
\forall k \exists i_o \quad \forall i \geq i_o \quad \frac{s_{k+1}}{\lambda(|r_k| - |r_{k+1}|)} \leq \frac{a_{m_{i+1}}}{b_{n_i}},
\]

since $s_{k+1} = \frac{1}{|r_k| - |r_{k+1}|} < \infty$

\[
g(s_{k+1}b_{n_k}) - g(s_kb_{n_k}) \leq g(s_{k+1}b_{n_k}) - g(s_{k+1}b_{n_{i+1}})
\leq g((|r_k| - |r_{k+1}|)\lambda a_{m_{i+1}})
\leq g(|r_k|\lambda a_{m_{i+1}}) - g(|r_{k+1}|\lambda a_{m_{i+1}})
= f(|r_k|a_{m_{i+1}}) - f(|r_{k+1}|a_{m_{i+1}})
\]

And by (i),
\[
\exists i_o, k_o \quad \forall i \geq i_o \text{ and } \forall k \geq k_o \quad \frac{a_{m_{i+1}}}{b_{n_{i+1}}} < \frac{s_k}{\lambda|r_k|}
\]

Hence
\[
f(|r_k|a_{m_{i+1}}) - f(|r_{k+1}|a_{m_{i+1}}) \leq f(|r_k|a_{m_{i+1}}) - f(|r_k|a_{m_{i+1}})
= g(\lambda |r_k| a_{m_{i+1}})
\leq g(s_kb_{n_{i+1}})
\leq g(s_{k+1}b_{n_{i+1}}) - g(s_kb_{n_{i+1}})
\]

Therefore (1) is a regular basis for $E \times F$.

3.3.3 $E = S_f(a, -1) \quad F = S_g(b, 1)$

Lemma 9:
If $(S_f(a, -1), S_g(b, 1)) \in (S^*)$, then
\[
\exists n_o, k \quad \forall K, m \quad \exists i_o, j_o \quad \forall i \geq i_o, \forall j \geq j_o
\]
(a) \( f(|r_m|a_i) < g((s_K - s_k)b_j) \Rightarrow f(|r_n_0|a_i) \leq g(s_Kb_j) \).

(b) \( g(s_kb_j) < f((|r_n_0| - |r_m|)a_i) \Rightarrow g(s_Kb_j) \leq f(|r_n_0|a_i) .\)

**Proof:** The same as the Lemma 8 in the section 3.3.2.

**Proof of the Theorem:**

(1) \( f \succ g : \ (S_f(a, -1), S_g(b, 1)) \in (S^*) \Rightarrow (a, f^{-1}g(b)) \in HP \not\subset 1 .\)

Similar to (1) in section 3.3.2.

(2) \( f \prec g : \ (S_f(a, -1), S_g(b, 1)) \in (S^*) \Rightarrow (g^{-1}f(a), b) \in HP \not\subset 1 .\)

It is the same as (2) of the section 3.3.2, except the conclusion. Since \( s_k \not\not\subset 1 \) we have \( (g^{-1}f(a), b) \in HP \not\subset 1 \) from

\[ \exists k \ \forall K \ s_k \leq A \Rightarrow s_K \leq A \]

For the regularity of the basis (1) we use

(i) \( \limsup_{i \to \infty} \frac{g^{-1}f(a_{m_i+1})}{b_{n_i+1}} < 1 \) and (ii) \( \liminf_{i \to \infty} \frac{g^{-1}f(a_{m_i+1})}{b_{n_i}} \geq 1 \)

instead of

(i) \( \sup_i \frac{g^{-1}f(a_{m_i+1})}{b_{n_i+1}} < \infty \) and (ii) \( \lim_{i \to \infty} \frac{g^{-1}f(a_{m_i+1})}{b_{n_i}} = \infty \)

in 3.3.2 and result follows similarly.

**3.3.4** \( E = S_f(a, 0) \quad F = S_g(b, 1) \)

**Lemma 10:**

If \( (S_f(a, 0), S_g(b, 1)) \in (S^*) \), then

\[ \exists n_0, k \ \forall K, m \ \exists i_o, j_o \ \forall i \geq i_o, \forall j \geq j_o : \]

(a) \( f(|r_m|a_i) < g((s_K - s_k)b_j) \Rightarrow f(|r_n_0|a_i) \leq g(s_Kb_j) .\)

(b) \( g(s_kb_j) < f((|r_n_0| - |r_m|)a_i) \Rightarrow g(s_Kb_j) \leq f(|r_n_0|a_i) .\)

**Proof:** It is the same as in the Lemma 8.
Proof of the Theorem:

(1) $f \succ g : (S_f(a,0), S_g(b,1)) \in (S^*) \Rightarrow (a, f^{-1}g(b)) \in HP.$
   Similar to the previous section 3.3.3.

(2) $f \prec g : (S_f(a,0), S_g(b,1)) \in (S^*) \Rightarrow (g^{-1}f(a), b) \in HP \not\cong 1.$
   Similar to the previous section 3.3.3.

(3) $f \approx g : (S_f(a,0), S_g(b,1)) \in (S^*) \Rightarrow (a, b) \in HP.$
   Similar to (3) in section 3.2.2, putting $|r_k| = s_{k+1} - s_k.$

Now we give the proof of

**Theorem 2:** Let $E = S_f(a, r)$, $F = S_g(b, s)$ and assume that $\text{Ext}(E \times F, E \times F) = 0$. Then $E \otimes \pi F$ has a regular basis in the following cases

(i) $f \prec g$ or $f \succ g.$

(ii) $f \approx g$ and $rs \not\cong -1.$

**Proof:** We only give the proof for $E = S_f(a, \infty)$, $F = S_g(b, \infty)$ and $f \prec g$ case, all the other cases being similar. $S_f(a, \infty) \otimes \pi S_g(b, \infty)$ is isomorphic to the Köthe space $\lambda(C)$ where $(C_{m,n}) = (e^f(r_k a_m) + g(s_k b_n)).$

We define $n_0 = 0$ and

$$I = \bigcup_{i=1}^{\infty} \{(m,n) : m_i + 1 \leq m \leq m_{i+1}, n \leq n_i\}.$$ 

and

$$J = \bigcup_{i=0}^{\infty} \{(m,n) : m \leq m_{i+1}, n_i+1 \leq n \leq n_{i+1}\}.$$ 

Then $I \cup J = \mathbb{N} \times \mathbb{N}, I \cap J = \emptyset.$ (See the figure.) Now we define a matrix $(D^k_{m,n})$ by

$$D^k_{m,n} = \begin{cases} 
ed^f(r_k a_m) & \text{if } (m,n) \in I \\ 
ed^g(s_k b_n) & \text{if } (m,n) \in J \end{cases}$$ 

First we show that the matrices $(C^k_{m,n})$ and $(D^k_{m,n})$ are equivalent. If $(m,n) \in I$, then there is a unique $i$ such that $m_i + 1 \leq m \leq m_{i+1}, n \leq n_i$.

$$g(s_k b_n) \leq g(s_k b_{n_i}) \leq f(r_k a_{m_i}, \ldots, r_k a_{m_{i+1}}) \leq f(r_k a_m) \leq g(s_k b_{n_{i+1}}).$$

(*) from subsection 3.1.1. So

$$C^k_{m,n} = e^f(r_k a_m) + g(s_k b_n) \leq e^f(r_{k+1} a_m) = D^k_{m,n}.$$
If \((m, n) \in J\), then there is a unique \(i\) such that \(m \leq m_{i+1}, \ n_{i+1} \leq n \leq n_{i+1}\). We have (**\textsuperscript{2}) from subsection 3.1.1.

\[
f(r_k a_m) \leq f(r_{k+1} a_{m+1}) \leq g(s_k b_{m+1}) \leq g(s_{k+1} b_n) - g(s_k b_n)
\]

Hence

\[
C_{m,n}^k = e^f(r_k a_m) + g(s_k b_n) \leq e^g(s_{k+1} b_n) = D_{m,n}^k
\]

\(D_{m,n}^k \leq C_{m,n}^k\) for all \(k, m, n\) is clear.

Finally we show that the matrix \((D_{m,n}^k)\) is regular when the elements \((m, n)\) are ordered as follows:

\[
(1, n_0 + 1), \ldots, (m_1, n_0 + 1), \quad (1, n_0 + 2), \ldots, (m_1, n_0 + 2), \ldots \quad \ldots, (1, n_1), \ldots, (m_1, n_1),
\]

\[
(m_1 + 1, 1), \ldots, (m_1 + 1, n_1), \quad (m_1 + 2, 1), \ldots, (m_1 + 2, n_1), \ldots \quad \ldots, (m_2, 1), \ldots, (m_2, n_1), \ldots
\]

\[
\ldots,
\]

\[
(1, n_{i-1} + 1), \ldots, (m_i, n_{i-1} + 1), \quad (1, n_{i-1} + 2), \ldots, (m_i, n_{i-1} + 2), \ldots \quad \ldots, (1, n_i), \ldots, (m_i, n_i),
\]

\[
(m_i + 1, 1), \ldots, (m_i + 1, n_i), \quad (m_i + 2, 1), \ldots, (m_i + 2, n_i), \ldots \quad \ldots, (m_{i+1}, 1), \ldots, (m_{i+1}, n_i),
\]

\[
(1, n_i + 1), \ldots, (m_{i+1}, n_i + 1), \quad (1, n_i + 2), \ldots, (m_{i+1}, n_i + 2), \ldots \quad \ldots, (1, n_{i+1}), \ldots, (m_{i+1}, n_{i+1}), \ldots
\]

Now we show the regularity of \((D_{m,n}^k)\) when we pass from \((m_i, n_i)\) to \((m_i + 1, 1)\):

\[
\frac{D_{m_i,n_i}^{k+1}}{D_{m_i,n_i}^k} \leq \frac{D_{m_{i+1},n_{i+1}}^{k+1}}{D_{m_{i+1},n_{i+1}}^k} \text{ is equivalent to } \frac{e^g(s_{k+1} b_{n_i})}{e^g(s_k b_{n_i})} \leq \frac{e^f(r_{k+1} a_{m_i+1})}{e^f(r_k a_{m_i+1})}
\]

and when we pass from \((m_{i+1}, n_i)\) to \((1, n_i + 1)\):

\[
\frac{D_{m_{i+1},n_i}^{k+1}}{D_{m_{i+1},n_i}^k} \leq \frac{D_{1,n_i+1}^{k+1}}{D_{1,n_i+1}^k} \text{ is equivalent to } \frac{e^f(r_{k+1} a_{m_i+1})}{e^f(r_k a_{m_i+1})} \leq \frac{e^g(s_{k+1} b_{n_i+1})}{e^g(s_k b_{n_i+1})}
\]

which have been shown before.
First we prove the following Lemma:

**Lemma:** Let $A$, $B = M(A)$, $F = m(B)$ be matrices with $A$ and $B$ non-negative. If $C$ is the $k$-th power of $A$, such that $m(C)$ is the least integer that is both a multiple of $m(A)$ and $m(B)$, then $m(C) = m(B)$. If $C$ is the $k$-th power of $B$, such that $m(C)$ is the least integer that is both a multiple of $m(A)$, then $m(C) = m(A)$.
Chapter 4

A Result On The Pseudo-Regularity of Köthe Spaces

First we prove the following Lemma:

**Lemma**: Let $E = \lambda(A)$, $F = \lambda(B)$ be two Köthe spaces. If $\Ext(E \times F, E \times F) = 0$, then there exist increasing sequences $(m(k))_k$, $(n(k))_k$, $(S_k)_k$, $(C_k)_k$ such that $\forall k, i, j$ we have

\[
\frac{S_{k+2} a_i^{m(k+2)}}{C_{k+1} b_j^{n(k+1)}} \leq \max \left\{ \frac{S_{k+3} a_i^{m(k+3)}}{C_{k+2} b_j^{n(k+2)}}, \frac{S_{k+1} a_i^{m(k+1)}}{C_k b_j^{n(k)}} \right\}
\]

(4.1)

and

\[
\frac{C_{k+2} b_j^{n(k+2)}}{S_{k+1} a_i^{m(k+1)}} \leq \max \left\{ \frac{C_{k+3} b_j^{n(k+3)}}{S_{k+2} a_i^{m(k+2)}}, \frac{C_{k+1} b_j^{n(k+1)}}{S_k a_i^{m(k)}} \right\}
\]

(4.2)

Moreover, if $F$ has the property $(DN)$, then we can choose $(n(k))_k$ and $(C_k)_k$ in such a way that they satisfy (4.1) and (4.2) and also

\[
(C_k b_j^{n(k)})^2 \leq (C_1 b_j^{n(1)})(C_{k+1} b_j^{n(k+1)}).
\]

(4.3)

**Proof**: First we observe that, if for a given $\mu$, $n_o$ and $k$ satisfy $(S^*)_o$, then for the same $\mu$, $(S^*)_o$ is satisfied for larger values of $n_o$ and $k$. We determine $(m(k))_k$, $(n(k))_k$, $(S_k)_k$, $(C_k)_k$ inductively. Put $m(1) = n(1) = 1$ and $C_i = S_i = 1$ where $i = 1, 2, 3$. For $\mu = n(1)$ in $(S^*)_1$ and $\bar{\mu} = m(1)$ in $(S^*)_2$ there exist $n_o, k, \bar{n}_o, \bar{k}$. We put $n(2) = \max\{k, \bar{n}_o, n(1) + 1\}$ and $m(2) = \max\{n_o, \bar{k}, m(1) + 1\}$. For $\mu = n(2)$ and $\bar{\mu} = m(2)$, there exist again $n_o, k, \bar{n}_o, \bar{k}$. We put $n(3) = \max\{k, \bar{n}_o, n(2) + 1\}$ and $m(3) = \max\{n_o, \bar{k}, m(2) + 1\}$.
Now in \((S^*)_1\), put \(\mu = n(1)\). By the above observation \(n_o = m(2)\), \(k = n(2)\) satisfy \((S^*)_1\). Taking \(K = n(3)\), \(m = m(3)\) and \(R = 1\) there exist \(\tilde{n} = \tilde{n}_4\) and \(\tilde{S} = \tilde{S}_4\). In \((S^*)_2\), put \(\mu = m(1)\) then we have \(n_o = n(2), k = m(2)\) taking \(K = m(3), m = n(3)\) and \(R = 1\) there exist \(\tilde{n} = \tilde{n}_4\) and \(\tilde{S} = \tilde{S}_4\). For \(\mu = n(3)\) and \(\mu = m(3)\) there exist \(n_o, k, \tilde{n}_o, \tilde{k}\). We put \(n(4) = \max\{k, \tilde{n}_o, n(3) + 1, \tilde{n}_4\}\), \(m(4) = \max\{n_o, \tilde{k}, m(3) + 1, \tilde{n}_4\}\) and \(S_4 = \tilde{S}_4S_3C_3, C_4 = \tilde{S}_4S_3C_3\).

Let \(m(1), \ldots, m(k + 2)\); \(n(1), \ldots, n(k + 2)\); \(S_1, \ldots, S_{k+2}\); \(C_1, \ldots, C_{k+2}\) be determined in this way.

Now, for \(\mu = n(k)\) we have \(n_o = m(k + 1), k = n(k + 1)\). We apply \((S^*)_1\) to \(K = n(k + 2), m = m(k + 2), R = \frac{S_{k+2}C_k}{C_{k+1}S_{k+1}}\) and obtain \(\tilde{n} = \tilde{n}_{k+3}, \tilde{S} = \tilde{S}_{k+3}\).

For \(\mu = m(k)\), we have \(n_o = n(k + 1), k = m(k + 1)\), we apply \((S^*)_2\) to \(K = m(k + 2), m = n(k + 2), R = \frac{C_{k+2}S_k}{C_{k+1}S_{k+1}}\) and obtain \(\tilde{n} = \tilde{n}_{k+3}\) and \(\tilde{S}_{k+3}\). To find \(m(k + 3)\) and \(n(k + 3)\), we again consider \(\mu = n(k + 2)\) and \(\mu = m(k + 2)\), then there exist \(n_o, k, \tilde{n}_o, \tilde{k}\). We put \(n(k + 3) = \max\{k, \tilde{n}_o, n(k + 2) + 1, \tilde{n}_{k+3}\}\) and \(m(k + 3) = \max\{n_o, \tilde{k}, m(k + 2) + 1, \tilde{n}_{k+3}\}\) such that
\[
\frac{a_i^{m(k+2)}}{b_j^{n(k+1)}} \leq \max \left\{ \frac{\tilde{S}_{k+3}a_i^{m(3)}}{b_j^{n(k+2)}}, \frac{C_{k+1}S_{k+1}a_i^{m(k+1)}}{C_kS_{k+2}b_j^{n(k)}} \right\}
\]
and
\[
\frac{b_j^{n(k+2)}}{a_i^{m(k+1)}} \leq \max \left\{ \frac{\tilde{S}_{k+3}b_j^{n(k+3)}}{a_i^{m(k+2)}}, \frac{C_{k+1}S_{k+1}b_j^{n(k+1)}}{C_kS_{k+2}a_i^{m(k)}} \right\}.
\]

We put \(S_{k+3} = \frac{\tilde{S}_{k+3}S_{k+2}C_{k+2}}{C_{k+1}}\) and \(C_{k+3} = \frac{\tilde{S}_{k+3}C_{k+2}S_{k+2}}{S_{k+1}}\). Hence we have (4.1) and (4.2).

Finally, when \(F\) has \((DN)\), we may choose
\[
n(k + 3) \geq \max\{k, \tilde{n}_o, n(k + 2) + 1, \tilde{n}_{k+3}\}\quad \text{and} \quad C_{k+3} \geq \frac{\tilde{S}_{k+3}C_{k+2}S_{k+2}}{S_{k+1}}
\]
such that both (4.2) and (4.3) hold.

Now we can prove the main result of this chapter :

**Theorem:** Let \(E\) and \(F\) be Schwartz Köthe spaces where \(E\) is regular and \(F\) has property \((DN)\). If \(\text{Ext}(E \times F, E \times F) = 0\) then \(E \times F\) and \(E \otimes_n F\) have pseudo-regular bases.
Proof : Let \( A = (a^+_i) \) and \( B = (b^-_j) \) be the Köthe matrices for \( E, F \), respectively. We choose sequences according to the above Lemma and use \((S_{k}a^m_i), (C_{k}b^n_j)\) which are equivalent to \( A \) and \( B \) respectively. Then we have (4.1) and (4.2). By using Proposition 1.5. in [8], (4.1) gives

\[
\frac{S_{k+2}a^m_i}{C_{k+2}b^n_j} \leq \frac{S_{k+3}a^m_i}{C_{k+2}b^n_j} \quad \text{for some } k \quad \Rightarrow 
\]

(4.4)

\[
\frac{S_{l+2}a^m_i}{C_{l+2}b^n_j} \leq \frac{S_{l+3}a^m_i}{C_{l+2}b^n_j} \quad \text{for all } l \geq k.
\]

Similarly from (4.2) we get

\[
\frac{C_{k+2}b^n_j}{S_{k+2}a^m_i} \leq \frac{C_{k+3}b^n_j}{S_{k+2}a^m_i} \quad \text{for some } k \quad \Rightarrow 
\]

(4.5)

\[
\frac{C_{l+2}b^n_j}{S_{l+2}a^m_i} \leq \frac{C_{l+3}b^n_j}{S_{l+2}a^m_i} \quad \text{for all } l \geq k.
\]

Since \( E \) and \( F \) are Schwartz spaces, we can find increasing sequences of indices \( (m_i) \) and \( (n_i) \) such that

\[
\frac{S_{i+2}a^{m(2)}_{m_{i+1}}}{C_{i}b^{n(1)}} \leq \frac{S_{i+3}a^{m(2)}_{m_{i+1}}}{S_{i+2}a^{m(2)}_{m_{i+1}}} \leq \ldots \leq \frac{S_{i}a^{m(3)}_{m_{i+1}}}{S_{i+2}a^{m(2)}_{m_{i+1}}} \leq \frac{C_{i}b^{n(2)}_{n_{i+1}}}{C_{i+2}b^{n(1)}} \leq \ldots
\]

(4.6)

If \( (e_i) \) and \( (f_j) \) denote the canonical bases for \( E \) and \( F \) respectively, we have that

\[
(x_n) = (\ldots, e_{m_i+1}, \ldots, e_{m_j+1}, f_{n_{j+1}}, \ldots)
\]

is a pseudo-regular basis for \( E \times F \). To see this, let \( m < n, m, n \in \mathbb{N} \). The following four cases are possible :

1. \( x_m = e_r, x_n = f_s \) with \( m_i + 1 \leq r \leq m_{i+1}, n_j + 1 \leq s \leq n_{j+1}, i \leq j \).

Then \( r < s \) and we have

\[
\frac{S_{3}a^{m(3)}_{r}}{S_{2}a^{m(2)}_{r}} \leq \frac{C_{2}b^{n(2)}_{s}}{C_{1}b^{n(1)}_{s}} \leq \frac{C_{4}b^{n(4)}_{s}}{C_{3}b^{n(3)}_{s}}
\]

and so by (4.5), we get

\[
\frac{S_{k+1}a^{m(k+1)}_{r}}{S_{k}a^{m(k)}_{r}} \leq \frac{C_{k+2}b^{n(k+2)}_{s}}{C_{k+1}b^{n(k+1)}_{s}} \quad \forall k \geq 2.
\]
(2) \( x_m = e_r, x_n = e_s \) with \( m_i + 1 \leq r \leq m_{i+1}, m_j + 1 \leq s \leq m_{j+1}, i \leq j \).
This case follows from the regularity of the matrix \((S_k a_i^{m(k)})\).

(3) \( x_m = f_r, x_n = e_s \) with \( n_{j-1} + 1 \leq r \leq n_j, m_i + 1 \leq s \leq m_{i+1}, j \leq i \).
In this case we have
\[
\frac{C_2 b_r^{m(2)}}{C_1 b_r^{m(1)}} \leq \frac{S_3 a_s^{m(3)}}{S_2 a_s^{m(2)}}
\]
Then using (4.4), we get
\[
\frac{C_{k+1} b_r^{n(k+1)}}{C_k b_r^{n(k)}} \leq \frac{S_{k+2} a_s^{m(k+2)}}{S_{k+1} a_s^{m(k+1)}} \quad \forall k \geq 1.
\]

(4) \( x_m = f_r, x_n = f_s \) with \( n_i + 1 \leq r \leq n_i, n_j + 1 \leq s \leq n_j, i \leq j \). This case also follows from the regularity of the matrix \((C_k b_i^{n(k)})\).

Hence these four cases show that \( E \times F \) has a pseudo-regular basis.

Next we consider \( E \overset{\wedge}{\otimes} s F \). We set
\[
A_i^k = \frac{S_{k+1} a_i^{m(k+1)}}{S_2 a_i^{m(2)}}, \quad B_j^k = \frac{C_{k+1} b_j^{n(k+1)}}{C_1 b_j^{n(1)}}, \quad k \geq 1.
\]
The matrices \((A_i^k)\) and \((B_j^k)\) are regular and
\[
A_i^1 = 1, A_i^k \geq 1 \quad \forall i, k, B_j^1 = 1, B_j^k \geq 1 \quad \forall j, k \text{ and } (B_j^k)^2 \leq (B_j^1)(B_j^{k+1}) = B_j^{k+1}.
\]
Then \( E \overset{\wedge}{\otimes} s F \) is isomorphic to the Köthe space \( \lambda(C) \) where \((C_{m,n}^k) = (A_m^k, B_n^k)\). We define \( n_o = 0 \) and
\[
I = \bigcup_{i=1}^{\infty} \{(m,n) : m_i + 1 \leq m \leq m_{i+1}, n \leq n_i\}.
\]
and
\[
J = \bigcup_{i=0}^{\infty} \{(m,n) : m \leq m_{i+1}, n_i + 1 \leq n \leq n_{i+1}\}.
\]
Then \( I \cup J = \mathbb{N} \times \mathbb{N}, I \cap J = \emptyset \). (See the figure) Now we define a matrix \((D_{m,n}^k)\) by
\[
D_{m,n}^k = \begin{cases} A_m^k & \text{if } (m,n) \in I \\
B_n^k & \text{if } (m,n) \in J
\end{cases}
\]
First we show that the matrices \((C_{m,n}^k)\) and \((D_{m,n}^k)\) are equivalent. If \((m,n) \in I\), then there is a unique \( i \) such that \( m_i + 1 \leq m \leq m_{i+1}, n \leq n_i \).
Then
\[ \frac{B_{n_i}^{k+1}}{B_{n_i}^k} = \frac{C_{k+1} b_{n_i}^{n(k+1)}}{C_k b_{n_i}^{n(k)}} \leq \frac{S_{k+2} a_{m_i+1}^{m(k+2)}}{S_{k+1} a_{m_i+1}^{m(k+1)}} = \frac{A_{m_i+1}^{k+1}}{A_{m_i+1}^k} \]
where (*) follows from (4.6) and (4.4). So
\[ B_n^k \leq B_{n_i}^k \leq \frac{B_{n_i}^{k+1}}{B_{n_i}^k} \leq \frac{A_{m_i+1}^{k+1}}{A_{m_i+1}^k} \leq \frac{A_{m}^{k+1}}{A_{m}^k} \]
from which it follows that 
\[ C_{m,n}^k = B_n^k A_{m}^k \leq A_{m}^{k+1} = D_{m,n}^{k+1} \]. If \((m, n) \in J\), then there is a unique \(i\) such that \(m \leq m_{i+1}, n_{i+1} \leq n \leq n_{i+1}\). We have
\[ \frac{S_3 a_{m_i+1}^{m(3)}}{S_2 a_{m_i+1}^{m(2)}} \leq \frac{C_3 b_{n_i+1}^{n(3)}}{C_2 b_{n_i+1}^{n(2)}} \leq \frac{C_4 b_{n_i+1}^{n(4)}}{C_3 b_{n_i+1}^{n(3)}} \leq \frac{C_4 b_{n_i+1}^{n(4)}}{S_3 a_{m_i+1}^{m(3)}}. \]
So by (4.5), we have
\[ \frac{C_{k+1} b_{n_i+1}^{n(k+1)}}{S_{k} a_{m_i+1}^{m(k)}} \leq \frac{C_{k+2} b_{n_i+1}^{n(k+2)}}{S_{k+1} a_{m_i+1}^{m(k+1)}} \quad \text{for all } k \geq 2. \]

Then
\[ \frac{A_{m_{i+1}}^k}{A_{m_{i+1}}^1} = \frac{S_{k+1} a_{m_{i+1}}^{m(k+1)}}{S_{k} a_{m_{i+1}}^{m(k)} - \frac{S_{k+1} a_{m_{i+1}}^{m(k+1)}}{S_{k+2} a_{m_{i+1}}^{m(2)}} \leq \frac{C_{k+2} b_{n_i+1}^{n(k+2)}}{C_{k+1} b_{n_i+1}^{n(k+1)}} \leq \frac{C_3 b_{n_i+1}^{n(3)}}{C_3 b_{n_i+1}^{n(3)}} = \frac{B_{n_i+1}^{k+2}}{B_{n_i+1}^3}. \]
Hence
\[ A_{m}^k = A_{m}^1 \leq \frac{A_{m_{i+1}}^k}{A_{m_{i+1}}^1} \leq \frac{B_{n_i+1}^{k+2}}{B_{n_i+1}^3} \leq \frac{B_{n}^{k+2}}{B_{n}^3} \leq \frac{B_{n}^{2k-1}}{B_{n}^k} \]
from which it follows that \[ C_{m,n}^k \leq B_{n}^{2k-1} = D_{m,n}^{2k-1}. \]

\[ D_{m,n}^k \leq C_{m,n}^k \] for all \(k, m, n\) is obvious.

Finally we show that the matrix \((D_{m,n}^k)\) is pseudo-regular when the elements \((m, n)\) are ordered as follows:
(1, n₀ + 1), ..., (m₁, n₀ + 1),  (1, n₀ + 2), ..., (m₁, n₀ + 2), ...
    ..., (1, n₁), ..., (m₁, n₁),

(m₁ + 1, 1), ..., (m₁ + 1, n₁),  (m₁ + 2, 1), ..., (m₁ + 2, n₁), ...
    ..., (m₂, 1), ..., (m₂, n₁), ...
    ...

(1, nᵢ₋₁ + 1), ..., (mᵢ, nᵢ₋₁ + 1),  (1, nᵢ₋₁ + 2), ..., (mᵢ, nᵢ₋₁ + 2),
    ..., (1, nᵢ), ..., (mᵢ, nᵢ),

(mᵢ + 1, 1), ..., (mᵢ + 1, nᵢ),  (mᵢ + 2, 1), ..., (mᵢ + 2, nᵢ), ...
    ..., (mᵢ₊₁, 1), ..., (mᵢ₊₁, nᵢ),

(1, nᵢ + 1), ..., (mᵢ₊₁, nᵢ + 1),  (1, nᵢ + 2), ..., (mᵢ₊₁, nᵢ + 2), ...
    ..., (1, nᵢ₊₁), ..., (mᵢ₊₁, nᵢ₊₁), ...

Let the ordered pair (r, u) appear before the ordered pair (s, v) in the above ordering. We have four cases:

1. (r, u) ∈ I,  (s, v) ∈ I
2. (r, u) ∈ I,  (s, v) ∈ J
3. (r, u) ∈ J,  (s, v) ∈ I
4. (r, u) ∈ J,  (s, v) ∈ J.

Now we show the pseudo-regularity of \( (D^k_{m,n}) \) in each case:

1. \( mᵢ + 1 \leq r \leq mᵢ₊₁, u \leq nᵢ, mⱼ + 1 \leq s \leq mⱼ₊₁, v \leq nⱼ, i \leq j \). Then \( r \leq s \).

Then by regularity of \( (A^k_r) \),

\[
\frac{A^{k+1}_{r}}{A^{k}_{r}} \leq \frac{A^{k+1}_{s}}{A^{k}_{s}} \quad \text{which is equivalent to} \quad \frac{D^{k+1}_{r,u}}{D^{k}_{r,u}} \leq \frac{D^{k+1}_{s,v}}{D^{k}_{s,v}}.
\]

2. \( mᵢ + 1 \leq r \leq mᵢ₊₁, u \leq nᵢ, s \leq mⱼ₊₁, nⱼ + 1 \leq v \leq nⱼ₊₁, i \leq j \), so \( r < v \).
As we have shown before, we have
\[
\frac{D_{r,u}^{k+1}}{D_{r,u}^k} = \frac{A_{r}^{k+1}}{A_r^k} \leq \frac{B_{u}^{k+2}}{B_{u}^{k+1}} = \frac{D_{s,v}^{k+2}}{D_{s,v}^{k+1}}.
\]

(3) \( r \leq m_i, n_{i-1} + 1 \leq u \leq n_i, v \leq n_j, m_j + 1 \leq s \leq m_{j+1}, i \leq j. \) Then \( u \leq s. \) So we have
\[
\frac{D_{r,u}^{k+1}}{D_{r,u}^k} = \frac{B_{u}^{k+1}}{B_u^k} \leq \frac{A_{s}^{k+2}}{A_{s}^{k+1}} = \frac{D_{s,u}^{k+2}}{D_{s,u}^{k+1}}.
\]

(4) \( m_i + 1 \leq r \leq m_{i+1}, u \leq n_i, s \leq m_{j+1}, n_j + 1 \leq v \leq n_{j+1}, i \leq j. \) This case can be shown as case (1), it follows from the regularity of \((B_t^k).\)

So for all cases we have \( \frac{D_{r,u}^{k+1}}{D_{r,u}^k} \leq \frac{D_{s,v}^{k+2}}{D_{s,v}^k}. \)
Chapter 5

Conclusion

The following is an interesting question in the theory of nuclear Köthe spaces: What are the conditions under which the vanishing of $\text{Ext}(E, E)$ implies that the Köthe space $E$ has a regular (or pseudo-regular) basis?

We showed that if $\text{Ext}(E \times F, E \times F) = 0$, then $E \times F$ and $E \hat{\otimes}_\pi F$ have regular bases, when $E = S_f(a, r)$, $F = S_g(b, s)$ and $f$ and $g$ are comparable with each other, except one case in which our method of proof did not work. However, we believe that it may be done by a different approach. But the question we stated at the beginning is still an open question in its most general form.

In most cases, the existence of a pseudo-regular basis is also strong enough to obtain almost all of the results that can be obtained using regularity. Since the choice of matrix representation of the basis is irrelevant in the use of pseudo-regularity, it is more convenient to work with than regularity.

Related with this subject, we also showed that when $E$ and $F$ are Schwartz Köthe spaces with $E$ regular, $F$ with property $(DN)$, $\text{Ext}(E \times F, E \times F) = 0$ implies $E \times F$ and $E \hat{\otimes}_\pi F$ have pseudo-regular bases.
REFERENCES


